Solvability and Weak Controllability of Fractional Degenerate Singular Problem

Achab Fatma [1*], Iqbal Batiha[2,3], Rezzoug Imad [4], Oussaief Takieddine [5], Adel Ouannas [6]
[1,4,5,6] Department of Mathematics and Computer Science, University of Oum El Bouaghi, Oum El Bouaghi 04000, Algeria
[2] Department of Mathematics, Al Zaytoonah University of Jordan, Amman 11733, Jordan
[3] Nonlinear Dynamics Research Center (NDRC), Ajman University, Ajman, UAE
Email: [1] achab.fatma@univ-oeb.dz, [2] i.batiha@zuj.edu.jo, [4] imad.rezzoug@univ-oeb.dz, [5] taki_maths@live.fr, [6] ouannas.adel@univ-oeb.dz
*Corresponding Author

***Abstract*—In this paper, our objective is to investigate the unique solvability and the weak controllability of the fractional degenerate and singular problem. The energy inequality method is gives a sufficient conditions for the existence and the uniqueness of the strong solution of our problem. This problem is ill-posed in the sense of Hadamard. To address this, we attempt regularization through a fractional Tikhonov regularization method, which not only establishes weak controllability but also provides a full characterization of the optimal control.**

***Keywords*—Fractional Degenerate Singular Problem, Energy Inequality Method, Weak Controllability, Tikhonov Regularization.**

I. Introduction

The modern fashion in scientific research in mathematics is the fractional differential equations (FDEs) that brought many scientists to follow this fashion, the importance and the success of fractional calculus lies in the accurate result in modeling, almost phenomena in science and engineering phenomena, also is a key tool to preserve the memory of processes and materials [1]–[5]. The paper [6] provides information on the mapping properties associated with different forms of fractional integration operators. In physics, FDEs can be used to model anomalous diffusion, where the movement of particles does not follow classical diffusion laws. For instance, in the study of transport phenomena in porous media, FDEs can provide a more accurate representation of the system's behavior by incorporating memory effects. To get more overview about fractional calculus and its implementations, the reader may refer to the references [7]–[17].

In other side, a controllability system is the ability to transform a system from a particular state to a desirable state exactly or approximately. Designing effective control strategies in engineering applications is crucial for example, consider an unmanned aerial vehicle (UAV) that needs to follow a specific trajectory. The UAV's motion can be modeled using a set of differential equations, and controllability ensures that the control inputs (e.g., rotor speeds) can be manipulated to guide the UAV along the desired path. This is crucial in applications like surveillance, where precise control is necessary to cover specific areas. However, to get more overview about controllability systems and their implementations, the reader may refer to the references [18]–[34].

There are many studies published in exact controllability also in null controllability and weak controllability [35]–[37]. Like this papers [38], [39] where the authors applied the notion of exact controllability to solve a semilinear abstract system and a singular degenerate wave equation. The authors of this paper [40] are used the null controllability method to solve some hyperbolique equation depending on a missing parameters, and this researchers [41]–[43] addressed to the weak controllability of some evolution systems.

Controllability of fractional differential equation also become an important topic until now, there are many literatures studies the controllability of different phenomena modeling by fractional differential equation for instance we cite [44], [45], [56]. The present paper is devoted to studying the solvability and the weak controllability of ill-posed singular and degenerate linear fractional parabolic problem with Dirichlet boundary condition. Anyhow, to get more overview about the solvability of fractional differential equations, the reader may refer to the references [46]–[55].

There are some challenges associated with ill-posed problem including this: ill-posed problem may have multiple solutions or no solution at all. Small changes or perturbations in the input data can lead to large variations in the solution, making the problem unstable. When solving ill-posed problems numerically, the algorithms employed may be prone to amplifying errors. Numerical instability can lead to unreliable results, making it challenging to trust the accuracy of the solution. Addressing these challenges requires careful consideration of the problem formulation for this we will used Tikhonov regularization.

The key tools used to study the existence and the uniqueness of the solution is the energy-inequality method which is based to find an a priori estimate. The weak controllability is achieved by a Tikhonov regularization method. Since the last method is applied to regularize the ill-posed problem which does not verify all the assumptions of the well-posed problem on the Hadamard sense which are the existence, uniqueness, and stability related to the given data, if one of these conditions is not satisfied comes an ill-posed problem.

The paper is divided into two parts, we start the first part by preliminaries, when we give some definitions and theorems on fractional calculus differentiation and integration which we needed in our studies, then we present the energy estimate method to solve the question of the existence and uniqueness of our problem when we start by searching for the a priori estimate which gives the uniqueness easily, moreover by the density of the range operator we prove the existence of the strong solution. The second part aims to study the weak controllability of the fractional problem to this aim we apply the Tikhonov regularization which also gives a full characterization of the control.

II. Preliminaries

In this section, we give some basic notions on fractional differentiation and integration which are necessary in our studies.

Definition 1: [57], [58] For $\alpha \in \mathbb{R}_+$, we define the fractional integral of a function $f \in L^1([0,T], X)$ of order α as follows:

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

where Γ is the Gamma function.

Definition 2: [57], [58] Let $0 < \alpha < 1$ and $f \in L^1([0,T], X)$. We define

1) The Left Caputo Derivative as

$$^cD_t^\alpha f = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s) ds.$$

2) The Right Caputo Derivatives as

$$_t^cD^\alpha f = \frac{-1}{\Gamma(1-\alpha)} \int_t^T (s-t)^{-\alpha} f'(s) ds.$$

3) The Left Riemann-Liouville Derivative as

$$^RD_t^\alpha f = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t (t-s)^{-\alpha} f(s) ds.$$

4) The Right Riemann-Liouville Derivative as

$$_t^RD^\alpha f = \frac{-1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_t^T (s-t)^{-\alpha} f(s) ds.$$

The right Caputo and Riemann-Liouville derivatives are connected by the following relationship:

$$^RD_t^\alpha f = ^c D_t^\alpha f + \frac{f(0)}{\Gamma(1-\alpha) t^\alpha}.$$

If $f(0) = 0$, then Riemann-Liouville derivative and Caputo derivative are coincides, i.e.,

$$^RD_t^\alpha f = ^c D_t^\alpha f.$$

Definition 3: [57], [58] For any $\sigma > 0$, we define the following semi-norms:

$$\begin{aligned} |f|^2_{^\iota H^\sigma(\Omega)} &:= \left\| ^RD_t^\sigma f \right\|^2_{L^2(\Omega)}, \\ |f|^2_{^r H^\sigma(\Omega)} &:= \left\| _t^RD^\sigma f \right\|^2_{L^2(\Omega)}, \\ |f|^2_{^c H^\sigma(\Omega)} &:= \left| \frac{\left(^RD_t^\sigma f, _t^R D^\sigma f \right)^2_{L^2(\Omega)}}{\cos(\sigma\pi)} \right|^{\frac{1}{2}}, \end{aligned}$$

and the following norms:

$$\begin{aligned} \|f\|^2_{^\iota H^\sigma(\Omega)} &= \left(\|f\|^2_{L^2(\Omega)} + |f|^2_{^\iota H^\sigma(\Omega)} \right)^{\frac{1}{2}}, \\ \|f\|^2_{^r H^\sigma(\Omega)} &:= \left(\|f\|^2_{L^2(\Omega)} + |f|^2_{^r H^\sigma(\Omega)} \right)^{\frac{1}{2}}, \\ \|f\|^2_{^c H^\sigma(\Omega)} &:= \left(\|f\|^2_{L^2(\Omega)} + |f|^2_{^c H^\sigma(\Omega)} \right)^{\frac{1}{2}}, \end{aligned}$$

where the space $^\iota H_0^\sigma(\Omega)$ and $^r H_0^\sigma(\Omega)$ are the closure spaces of $C_0^\infty(\Omega)$ with respect to the norms $\|.\|^2_{^\iota H^\sigma(\Omega)}$ and $\|.\|^2_{^r H^\sigma(\Omega)}$, respectively.

Lemma 1: [57], [58] For any real $\sigma \in \mathbb{R}_+$, if $f \in ^\iota H^\sigma(\Omega)$ and $g \in C_0^\infty(\Omega)$, then

$$\left(^RD_t^\sigma f, g \right)_{L^2(\Omega)} = \left(f, _t^R D^\sigma g \right)_{L^2(\Omega)}.$$

Lemma 2: [57], [58] For $0 < \sigma < 2$ such that $\sigma \neq 1$ and $f \in H_0^{\frac{\sigma}{2}}(\Omega)$, we have

$$^RD_t^\sigma f = ^R D_t^{\frac{\sigma}{2}} {^R D_t^{\frac{\sigma}{2}}} f.$$

Lemma 3: [57], [58] For $\sigma \in \mathbb{R}_+$ such that $\sigma \neq n + \frac{1}{2}$, the semi-norms $|.|_{^\iota H^\sigma(\Omega)}$, $|.|_{^r H^\sigma(\Omega)}$ and $|.|_{^c H^\sigma(\Omega)}$ are equivalent. Then we pose

$$|.|_{^\iota H^\sigma(\Omega)} \cong |.|_{^r H^\sigma(\Omega)} \cong |.|_{^c H^\sigma(\Omega)}.$$

Lemma 4: For $\sigma > 0$, the space $^r H^\sigma(\Omega)$ with respect to the norm $\|.\|_{^r H^\sigma(\Omega)}$ is complete.

Theorem 1: [45] The mild solution $z = z(x,t,v) \in C([0,T], Y)$ of the problem (2) is given by

$$z(t) = S_\alpha(t) z_0 + \int_0^t (t-s)^{\alpha-1} P_\alpha(t-s) (f(s) - v(s)) ds,$$

for $t \in [0,T]$ for every $v \in L^2([0,T], U)$.

Let Φ_α be a Mainardi function defined as

$$\Phi_\alpha(z) = \sum_{n=0}^{+\infty} \frac{(-z)^n}{n!\Gamma(-\alpha n + 1 - \alpha)}.$$

We set

$$\begin{aligned} S_\alpha(t) &= \int_0^\infty \Phi_\alpha(\theta) R(\theta t^\alpha) d\theta, \\ S_\alpha^*(t) &= \int_0^\infty \Phi_\alpha(\theta) R^*(\theta t^\alpha) d\theta, \\ P_\alpha(t) &= \int_0^\infty \alpha\theta\Phi_\alpha(\theta) R(t^\alpha\theta) d\theta, \end{aligned}$$

where S_α and P_α verifying the assumptions in [45].

III. Position of the Problem

Studying the singular degenerate fractional problems is a relevant and important area in mathematics and applied sciences. Many physical systems exhibit fractional behavior, especially those with memory effects, and singular degenerate fractional models provide more accurate descriptions of their dynamics. Examples include viscoelastic materials, anomalous diffusion processes, and various biological and engineering systems.

We consider a degenerate and singular fractional parabolique controller problem, when the control fact in the second member of the equation. This problem describe the heat conduction in relation to the body geometric shape more details see [59], we want to study the weak controllability of our problem. We start our studies by the existence and the uniqueness of the strong solution to the problem (1), then we try to achieve the weak controllability of the problem.

Let Ω be an open bounded domain with smooth boundary Γ, we denoted by $Q = \Omega \times (0,T)$ and $\Sigma = \Gamma \times (0,T)$, where $\Omega = (0,l)$. We consider the following fractional controller for degenerate and singular problem:

$$\begin{aligned} {}_0^cD_t^\alpha y(x,t) - \frac{\partial}{\partial x}(x^\beta \frac{\partial}{\partial x} y(x,t)) + b(x,t)y(x,t) = \tilde{f} + v, &\text{in } Q \\ y(x,0) = \varphi(x), \text{ on } \Omega,& \\ y(0,t) = y(l,t) = 0, \text{ in } \Sigma,& \end{aligned} \tag{1}$$

where $0 < \alpha < 1$, $0 \le \beta < 1$, and ${}^cD^\alpha$ is the Caputo fractional derivative, whereas b, $\tilde{f}$ and φ are known functions, and v is the control function that belongs to the control space U for which the function φ satisfies the following compatibility condition:

$$\varphi(0) = \varphi(l) = 0,$$

and the function b verifies

$$0 < b_0 \le b(x,t) \le b_1.$$

Now, we assume that

$$z(x,t) = y(x,t) - \varphi(x),$$

which implies

$$y(x,t) = z(x,t) + \varphi(x).$$

Thus, problem (1) becomes

$$\begin{aligned} {}_0^cD_t^\alpha z(x,t) - \frac{\partial}{\partial x}(x^\alpha \frac{\partial}{\partial x} z(x,t)) + b(x,t)z(x,t) = f + v, &\text{in } Q \\ z(x,0) = 0, \text{ on } \Omega,& \\ z(0,t) = z(l,t) = 0, \text{ in } \Sigma,& \end{aligned} \tag{2}$$

where

$$f = \tilde{f} - {}_0^c D_t^\alpha \varphi(x) + \frac{\partial}{\partial x}(x^\alpha \frac{\partial}{\partial x}\varphi(x)) - b(x,t)\varphi(x).$$

The problem (2) can be then rewritten as follows:

$$Lz = \mathcal{F},$$

where $L = (\mathcal{L}, l)$ with the domain of definition E, while $\mathcal{F} = (f + v, 0)$ belongs to $F = L^2(Q)$ such that

$$\mathcal{L}z = {}_t^c D_t^\alpha z(x,t) - \frac{\partial}{\partial x}(x^\alpha \frac{\partial}{\partial x} z(x,t)) + b(x,t)z(x,t) = f + v,$$

with the initial condition

$$lz = z(x,0) = 0, \ \forall x \in (0,l).$$

We shall next study the existence and the uniqueness of strong solution to the controller problem (2).

IV. Solvability of the Fractional Controller System

This section concerned to study the strong solution of the problem (1) using energy estimate method. This method based to find an a priori estimate which is obtained by the help of an multiplicators which is specific for each problem. It is evident to conclude the uniqueness of the solution from the a priori estimation and also it's play a main role to prove the existence of the strong solution. Recently these scientific papers [57], [58] published to study the unique solvability of fractional problems.

A. A Priori Estimation

In this step, we choose the multiplicator $Mv(x,t) = v(x,t)$ to establish an a priori estimate in specific space to solve the problem. We consider

$$Lz = \mathcal{F},$$

where $L = (\mathcal{L}, l)$ with the domain of definition E such that ${}_0^cD^{\frac{\alpha}{2}}{}_t$, $\frac{\partial v}{\partial x} \in L^2(Q)$ and $\mathcal{F} = (f + v, 0)$.

Theorem 2: For any function $z \in E$, we have the inequality

$$\|z\|_E \le c\|Lz\|_{L^2(Q)}, \tag{3}$$

where c is a positive constant independent of z. The precedent estimation proves the uniqueness of the solution.

Proof: By multiplying the first equation in the problem (2) by the following operator:

$$Mz = z(x,t),$$

and then by integrating the result over Q, we obtain

$$\int\limits_Q \left({}_0^cD_t^\alpha z(x,t) - \frac{\partial}{\partial x}(x^\beta \frac{\partial}{\partial x} z(x,t)) + b(x,t)z(x,t) \right) \times z(x,t)dxdt = \int\limits_Q (f+v)\, z(x,t)dxdt.$$

Now, we have that $z(x,o)=0$, which implies that ${}_0^cD_t^\alpha = {}_0^R D_t^\alpha$. So, we can apply Lemmas (3) and (4) and Definition (3), we get

$$\begin{aligned}\int_Q {}_0^cD_t^\alpha z(x,t)z(x,t)dxdt &= \left({}_0^cD_t^{\frac{\alpha}{2}}\, {}_0^cD_t^{\frac{\alpha}{2}} z, z\right)_{L^2(Q)} \\ &= \left({}_0^cD_t^{\frac{\alpha}{2}} z, {}_t^cD_T^{\frac{\alpha}{2}} z\right)_{L^2(Q)} \\ &= \cos\left(\frac{\alpha\pi}{2}\right)|z|_{{}^cH^{\frac{\alpha}{2}}(Q)} \\ &\simeq \cos\left(\frac{\alpha\pi}{2}\right)|z|_{{}^lH^{\frac{\alpha}{2}}(Q)} \\ &= \cos(\frac{\alpha\pi}{2})\left\|{}_0^cD_t^{\frac{\alpha}{2}}\right\|^2_{L^2(Q)},\end{aligned}$$

and by integration by parts over $(0,l)$, we then obtain

$$-\int\limits_Q \left(\frac{\partial}{\partial x}(x^\beta \frac{\partial}{\partial x} z(x,t))\right) z(x,t)dxdt = \left\| x^{\frac{\beta}{2}} \frac{\partial z}{\partial x}\right\|^2_{L^2(Q)}.$$

Now, by replacing the two expressions in (IV-A), we get

$$\cos\left(\frac{\alpha\pi}{2}\right)\left\|{}_0^cD_t^{\frac{\alpha}{2}}\right\|^2_{L^2(Q)} + \left\| x^{\frac{\beta}{2}} \frac{\partial z}{\partial x}\right\|^2_{L^2(Q)} + (b_0-\epsilon)\|z\|^2_{L^2(Q)} \le \frac{1}{2\epsilon}\left(\|f\|^2_{L^2(Q)} + \|v\|^2_{L^2(Q)}\right).$$

Consequently, we obtain

$$\left\|{}_0^cD_t^{\frac{\alpha}{2}} z\right\|^2_{L^2(Q)} + \left\| x^{\frac{\beta}{2}} \frac{\partial z}{\partial x}\right\|^2_{L^2(Q)} + \|z\|^2_{L^2(Q)} \le C\left(\|f\|^2_{L^2(Q)} + \|v\|^2_{L^2(Q)}\right),$$

where

$$C = \frac{1}{2\epsilon}\left(\frac{1}{\min(\cos(\frac{\alpha\pi}{2}), 1, b_0-\epsilon)}\right).$$

Hence, we have

$$\|z\|_E \le c\,\|Lz\|_{L^2(Q)},$$

with

$$c = \sqrt{C}.$$

Let z_1 and z_2 be two distinguish solutions, then

$$\begin{cases} Lz_1 = \mathcal{F}, \\ Lz_2 = \mathcal{F}. \end{cases}$$

Due to the linearity of the operator L, we can have

$$L(z_1 - z_2) = 0,$$

which implies

$$\|z_1 - z_2\|_E \le c\,\|L(z_1-z_2)\|_{L^2(Q)} = 0.$$

This consequently gives

$$\|z_1 - z_2\|_E \le 0.$$

Finally, we can have

$$z_1 - z_2 = 0,$$

which proves the uniqueness of the solution. ■

Proposition 1: The operator L with the domain of definition $D(L)$ has a closure $\bar{L}$.

Proof: The main idea here for proving that the operator L admits a certain closure is to assume that there exists a sequence $(w_n)_{n\in\mathbb{N}} \subset D(L)$ such that

$$w_n \to 0 \text{ in } E, \tag{4}$$

and

$$Lw_n \to \mathcal{F} \text{ in } L^2(Q). \tag{5}$$

This immediately implies

$$f + v = 0.$$

From (4) and (5), we deduce

$$w_n \to 0 \text{ in } D'(Q),$$

and

$$Lw_n \to \mathcal{F} \text{ in } D'(Q).$$

Thank's to the continuity of the fractional derivation and the first order of the derivation from $D'(Q)$ to $D'(Q)$ that gives

$$Lw_n \to 0 \text{ in } D'(Q).$$

Due to the uniqueness of the limit in $D'(Q)$, we get

$$\mathcal{F} = 0.$$

It means that

$$f + v = 0.$$

■

Definition 4: We say that z is a strong solution to problem (2) iff z verifies the following equation:

$$\bar{L}z = \mathcal{F},$$

where $\bar{L}$ is the closure of L with the domain of definition $D(\bar{L})$. The priori estimate stays valid for the strong solution, i.e.,

$$\|z\|_E \leq c \left\|\bar{L}z\right\|_{L^2(Q)}.$$

We therefore deduce from the last estimation that the strong solution of (2) is unique, if it exists.

Corollary 1: The range $R(\bar{L})$ is equal to the closure of $R(L)$. It means that $R(\overline{L}) = \overline{R(L)}$.

Proof: The proof here can be performed in the same way of [57], [58]. ■

B. Existence of Solution

In this step, we aim to prove the existence of the strong solution to problem (2). To this end, it is sufficient to check the density of the operator $R(L)$.

Theorem 3: Let the estimation (3) be satisfied, then for all $\mathcal{F} = (f + v, 0) \in F = L^2(Q)$, the problem (2) admits a unique strong solution.

Proof: Let $P = (p, 0)$ such that $p \in R(L)^{\perp} \subset L^2(Q)$. For all $z \in E$, we can have

$$(Lz, P)_F = \int_Q z.pdxdt.$$

Now, we need to demonstrate that $R(L)^{\perp} = \{0\}$. For this purpose, we choose $p = z$ to get

$$\left\|{}_0^c D_t^{\frac{\alpha}{2}} z\right\|_{L^2(Q)}^2 + \left\|x^{\frac{\beta}{2}} \frac{\partial z}{\partial x}\right\|_{L^2(Q)}^2 + \|z\|_{L^2(Q)}^2 = 0.$$

This yields to (3), and so we have

$$\|z\|_E \leq 0.$$

Thus, we obtain that $p = z = 0$, and hence $\overline{R(L)} = F$. ■

V. Approximate controllability

Understanding the controllability of singular degenerate fractional problems is crucial for designing effective control strategies for systems described by such models. Optimal control aims to find the best control input to guide a system to a desired state while considering the constraints and dynamics of the system. We aims to study the weak controllability of the fractional degenerate and singular problem, this problem is ill-posed in the sense of Hadamard which requires us to use a Tikhonov regularization method also, it helps us to prove the weak controllability. The controller will be characterized by an optimality system.

The method of Tikhonov regularization for the control problem is constructed to solve the following problem:

$$\inf_{v \in L^2(Q)} J(v), \tag{6}$$

where

$$J(v) = \left\|I^{1-\alpha} z(T) - h\right\|_{L^2(Q)}^2 + \lambda \|v\|_{L^2(Q)}^2, \tag{7}$$

and where $z \in Y = L^2(Q)$, h is desired state that belongs to $L^2(Q)$, and λ is the regularized parameter.

Definition 5: We say that the problem of control (2) is approximate controllable on $[0, T]$ if

$$\overline{C(T, z_0, v)} = X,$$

where

$$C(T, z_0, v) = \left\{I^{1-\alpha} z(T, z_0, v),\ v \in L^2(Q)\right\}.$$

Theorem 4: For $\frac{1}{2} < \alpha < 1$, the problem (6) with (7) admits a unique optimal control.

Proof: Let a minimizing sequence (v_n) be such that

$$\inf_{v \in L^2(Q)} J(v) = \lim_{n \to +\infty} J(v_n) = m.$$

Moreover, we have

$$J(v_n) \leq m.$$

This implies that there exists a constant C such that

$$\begin{aligned} \left\|I^{1-\alpha} z_n(T) - h\right\|_{L^2(Q)} &\leq C, \\ \|v_n\|_{L^2(Q)} &\leq \lambda^{\frac{-1}{2}} C. \end{aligned}$$

So, there exist subsequences, denoted by (v_n) and $\left(I^{1-\alpha} z_n(T)\right)$, such that

$$\begin{aligned} I^{1-\alpha} z_n(T) &\to \theta \text{ in } L^2(Q), \\ v_n &\to u \text{ in } L^2(Q). \end{aligned}$$

Thus, we have

$$z_n(t) = S_\alpha(t) z_0 + \int_0^t (t-s)^{\alpha-1} P_\alpha(t-s) \left(g(s) + v_n(s)\right) ds,$$

where $t \in [0, T]$. For any function $\phi \in L^2(Q)$, we can then have

$$\begin{aligned} \langle z_n, \phi \rangle_{L^2(Q)} &= \langle S_\alpha(t) z_0, \phi \rangle_{L^2(Q)} \\ &\quad + \int_0^t (t-s)^{\alpha-1} \langle P_\alpha(t-s) g(s), \phi \rangle_{L^2(Q)} ds \\ &\quad + \int_0^t (t-s)^{\alpha-1} \langle v_n(s), P_\alpha^*(t-s) \phi \rangle_{L^2(Q)} ds. \end{aligned}$$

Now, passing to the limit in the previous expression yields

$$\lim_{n\to\infty}\langle z_n,\phi\rangle_{L^2(Q)} = \langle S_\alpha(t)z_0,\phi\rangle_{L^2(Q)} + \int_0^t (t-s)^{\alpha-1}\langle P_\alpha(t-s)\,g(s),\phi\rangle_{L^2(Q)}\,ds + \int_0^t (t-s)^{\alpha-1}\langle P_\alpha(t-s)\,u(s),\phi\rangle_{L^2(Q)}\,ds = \langle z,\phi\rangle_{L^2(Q)},$$

which means

$$z_n \to z \text{ in } L^2(Q).$$

According to Proposition 3.5 in [45], we can get

$$z_n \to z \text{ in } C([0,T],Y).$$

Also, we have $z_n \in C([0,T],Y)$. Then, we obtain $I^{1-\alpha}z_n \in C([0,T],Y)$, and consequently we have

$$\langle I^{1-\alpha}z_n(T),\phi\rangle_{L^2(Q)} = \frac{1}{\Gamma(1-\alpha)} \times \int_0^T (T-s)^{-\alpha}\langle S_\alpha(s)z_0,\phi\rangle_{L^2(Q)} + \frac{1}{\Gamma(1-\alpha)} \times \int_0^T\int_0^s \frac{(T-s)^{-\alpha}}{(s-\tau)^{1-\alpha}}\langle P_\alpha(s-\tau)\,g(\tau),\phi\rangle_{L^2(Q)}\,d\tau ds + \frac{1}{\Gamma(1-\alpha)}\int_0^T\int_0^s \frac{(T-s)^{-\alpha}}{(s-\tau)^{1-\alpha}}\langle v_n(\tau),P_\alpha^*(s-\tau)\,\phi\rangle_{L^2(Q)}\,d\tau ds.$$

By passing to limit, we get

$$\langle \theta,\phi\rangle_{L^2(Q)} = \frac{1}{\Gamma(1-\alpha)}\int_0^T (T-s)^{-\alpha}\langle S_\alpha(s)z_0,\phi\rangle_{L^2(Q)} + \frac{1}{\Gamma(1-\alpha)}\int_0^T\int_0^s \frac{(T-s)^{-\alpha}}{(s-\tau)^{1-\alpha}}\langle P_\alpha(s-\tau)\,g(\tau),\phi\rangle_{L^2(Q)}\,d\tau ds + \frac{1}{\Gamma(1-\alpha)}\int_0^T\int_0^s \frac{(T-s)^{-\alpha}}{(s-\tau)^{1-\alpha}}\langle P_\alpha(s-\tau)\,u(\tau),\phi\rangle_{L^2(Q)}\,d\tau ds = \langle I^{1-\alpha}z(T),\phi\rangle_{L^2(Q)}.$$

As a result, we deduce

$$I^{1-\alpha}z_n(T) \to I^{1-\alpha}z(T) \text{ in } L^2(Q).$$

Due to J is coercive and lower semi-continuous, we get

$$J(u) \le \lim_{n\to+\infty}\inf_{v\in L^2(Q)} J(v_n) = \inf_{v\in L^2(Q)} J(v) = m.$$

This implies that u is the optimal control corresponding to the associate optimal state z. The uniqueness of u can be then obtained from the strictly convexity of J. ■

A. Characterization Of the Optimal Control

This subsection is devoted to characterize the optimal control solution of the regularization problem via optimality systems, and also to give some necessary conditions to achieve the weak controllability. In fact, a first-order optimality condition for J can give

$$J'(u)(v-u) = 0,\ \forall v \in L^2(Q).$$

By performing simple calculations, we obtain

$$J'(u)(v-u) = \left\langle I^{1-\alpha}z(T)-h, I^{1-\alpha}\psi(T)\right\rangle_{L^2(Q)} + \lambda\langle u, v-v\rangle_{L^2(Q)} = 0,\ \forall v\in L^2(Q),$$

where $\psi(t)$ is solution of the following problem:

$$\begin{aligned} &{}_0^cD_t^\alpha\psi(x,t) - \frac{\partial}{\partial x}(x^\beta\frac{\partial}{\partial x}\psi(x,t)) + b(x,t)\psi(x,t) = v-u,\\ &\psi(x,0) = 0,\ \text{on } \Omega,\\ &\psi(0,t) = \psi(l,t) = 0,\ \text{in } \Sigma. \end{aligned} \tag{8}$$

Now, we can define an adjoint state $q = q(t,x,v)$ solution as follows:

$$\begin{cases} {}_t^cD_T^\alpha q - \frac{\partial}{\partial x}(x^\beta\frac{\partial}{\partial x}q) + b(x,t)q = 0,\ \text{in } Q\\ q(T) = I^{1-\alpha}z(T)-h,\ \text{on } \Omega,\\ q(0,t) = q(l,t) = 0,\ \text{in } \Sigma. \end{cases} \tag{9}$$

The solution of problem (9)is then given by

$$q(t) = S_\alpha^*(T-t)\left[I^{1-\alpha}z(T)-h\right], t\in[0,T].$$

By multiplying the first equation of (8) by q solution of (9), and by using Lemma 4, we can have

$$\langle I^{1-\alpha}z(T)-h, I^{1-\alpha}\psi(T)\rangle_{L^2(Q)} = \left\langle q(T), I^{1-\alpha}\psi(T)\right\rangle_{L^2(Q)} = \langle q, v-u\rangle_{L^2(Q)},$$

which implies

$$J'(u)(v-u) = \langle q, v-u\rangle_{L^2(Q)} + \lambda\langle u, u-v\rangle_{L^2(Q)} = 0,$$

for all $v\in L^2(Q)$. Consequently, we obtain

$$u = -\lambda^{-1}q = -\lambda^{-1}S_\alpha^*(T-t)\left[I^{1-\alpha}z(T)-h\right] \text{ a.e in } Q.$$

Then, the optimal control is characterized by the following optimality system:

$$\begin{aligned} &{}_0^cD_t^\alpha z(x,t) - \frac{\partial}{\partial x}(x^\beta\frac{\partial}{\partial x}z(x,t)) + b(x,t)z(x,t) = f+u,\\ &z(x,0) = 0,\ \text{on } \Omega,\\ &z(0,t) = z(l,t) = 0,\ \text{in } \Sigma.\\ &-D^\alpha q(x,t) - \frac{\partial}{\partial x}(x^\alpha\frac{\partial}{\partial x}q(x,t)) + b(x,t)q(x,t) = 0,\\ &q(x,T) = I^{1-\alpha}z(T)-h,\ \text{on } \Omega,\\ &q(0,t) = q(l,t) = 0,\ \text{in } \Sigma. \end{aligned}$$

with

$$u = -\lambda^{-1}q = -\lambda^{-1}S_\alpha^*(T-t)\left[I^{1-\alpha}z(T)-h\right] \text{ a.e in } Q.$$

Theorem 5: The problem (2) is weak controllable iff the operator $\lambda R(\lambda, -\Lambda_T)$ converges to zero when $\lambda \to 0$, where Λ_T is an operator defined from $L^2(Q)$ to $L^2(Q)$ by

$$\Lambda_T = \frac{1}{\Gamma(1-\alpha)} \int_0^T \int_0^s \frac{(T-s)^{-\alpha}}{(s-\tau)^{1-\alpha}} P_\alpha(s-\tau) S(s-\tau) d\tau ds,$$

and

$$R(\lambda, -\Lambda_T) = (\lambda I + \Lambda_T)^{-1}$$

such that $(\lambda I + \Lambda_T)$ is invertible.

Proof: We can have

$$I^{1-\alpha} z(T) = \frac{1}{\Gamma(1-\alpha)} \int_0^T (T-s)^{-\alpha} [S_\alpha(s) z_0 + r(s)] ds - \lambda^{-1} \Lambda_T \left[I^{1-\alpha} z(T) - h \right],$$

where

$$r(s) = \int_0^s (s-\tau)^{\alpha-1} P_\alpha(s-\tau) g(\tau) d\tau.$$

Consequently, it is easy to get

$$I^{1-\alpha} z(T) - h = \lambda R(\lambda, -\Lambda_T) \times \left(\frac{1}{\Gamma(1-\alpha)} \int_0^T (T-s)^{-\alpha} [S_\alpha(s) z_0 + r(s)] ds - h \right).$$

By using the Theorem 2 in [60], we can infer that if $\lambda R(\lambda, -\Lambda_T) \to 0$, when λ tends to 0, then we have

$$\overline{C(T, z_0, v)} = X,$$

which proves the approximate controllability. ■

VI. Conclusion

From the fact that the controllability of singular and degenerate fractional problem is an important result in the theory of FDE, we have discussed the solvability of a controllability problem using the energy estimate method. We have proved the weak controllability of the problem in question by Tikhonov regularization method. Then we have characterized the optimal control by an optimality system.

References

[1] I. M. Batiha, O. Talafha, O. Y. Ababneh, S. Alshorm, and S. Momani, "Handling a commensurate, incommensurate, and singular fractional-order linear time-invariant system," *Axioms*, vol. 12, no. 8, 2023, doi: 10.3390/axioms12080771.

[2] N. R. Anakira, O. Ababneh, A. S. Heilat, and I. Batiha, "A new accurate approximate solution of singular two-point boundary value problems," *General Letters in Mathematics*, vol. 12, no, 1, pp. 31–39, 2022, doi: 10.31559/glm2022.12.1.4.

[3] R. B. Albadarneh, A. K. Alomari, N. Tahat, and I. M. Batiha, "Analytic solution of nonlinear singular BVP with multi-order fractional derivatives in electrohydrodynamic flows," *Turkish World Mathematical Society Journal of Applied and Engineering Mathematics*, vol. 11, no. 4, pp. 1125–1137, 2021.

[4] I. M. Batiha, R. El-Khazali, A. AlSaedi, and S. Momani, "The general solution of singular fractional-order linear time-invariant continuous systems with regular pencils," *Entropy*, vol. 20, pp. 1–14, 2018, doi: 10.3390/e20060400.

[5] Z. Dahmani, A. Anber, and I. Jebril, "Solving conformable evolution equations by an extended numerical method," *Jordan Journal of Mathematics and Statistics*, vol. 15, no. 2, pp. 363–380, 2022, doi: 10.47013/15.2.14.

[6] A. A. Kilbas, O. I. Marichev, and S. G. Samko, "Fractional Integrals and Derivatives: Theory and Applications," *Gordon and Breach Science Publishers: Switzerland*, 1993.

[7] I. M. Batiha, N. Barrouk, A. Ouannas, and A. Farah, "A study on invariant regions, existence and uniqueness of the global solution for tridiagonal reaction-diffusion systems," *Journal of Applied Mathematics and Informatics*, vol. 41, no. 4, pp. 893–906, 2023, doi: 10.14317/jami.2023.893.

[8] I. Batiha, A. Ouannas, and J. Emwas, "A stabilization approach for a novel chaotic fractional-order discrete neural network," *Journal of Mathematical and Computational Science*, vol. 11, no. 5, pp. 5514–5524, 2021, doi: 10.28919/jmcs/6004.

[9] Z. Chebana, T. E. Oussaeif, S. Dehilis, A. Ouannas, and I. M. Batiha, "On nonlinear Neumann integral condition for a semilinear heat problem with blowup simulation," *Palestine Journal of Mathematics*, vol. 12, no. 3, pp. 378–394, 2023.

[10] R. B. Albadarneh, A. Abbes, A. Ouannas, I. M. Batiha, and T. E. Oussaeif, "On chaos in the fractional-order discrete-time macroeconomic systems," *AIP Conference Proceedings*, vol. 2849, no. 1, 2023, doi: 10.1063/5.0162686.

[11] I. M. Batiha, N. Djenina, and A. Ouannas, "A stabilization of linear incommensurate fractional-order difference systems," *AIP Conference Proceedings*, vol. 2849, no. 1, 2023, doi: 10.1063/5.0164866.

[12] N. Abdelhalim, A. Ouannas, I. Rezzoug, and I. M. Batiha, "A study of a high–order time–fractional partial differential equation with purely integral boundary conditions," *Fractional Differential Calculus*, vol. 13, no. 2, pp. 199–210, 2023, doi: 10.7153/fdc-2023-13-13.

[13] A. A. Khennaoui, A. O. Almatroud, A. Ouannas, M. M. Al-sawalha, G. Grassi, V. T. Pham, and I. M. Batiha, "An unprecedented 2-dimensional discrete-time fractional-order system and its hidden chaotic attractors," *Mathematical Problems in Engineering*, vol. 2021, pp. 1–10, 2021, doi: 10.1155/2021/6768215.

[14] I. M. Batiha, N. Allouch, I. H. Jebril, and S. Momani, "A robust scheme for reduction of higher fractional-order systems," *Journal of Engineering Mathematics*, vol. 144, no. 4, 2024, doi: 10.1007/s10665-023-10310-6.

[15] I. M. Batiha, A. Benguesmia, T. E. Oussaeif, I. H. Jebril, A. Ouannas, and S. Momani, "Study of a superlinear problem for a time fractional parabolic equation under integral over-determination condition," *IAENG International Journal of Applied Mathematics*, vol. 54, pp. 187–193, 2024.

[16] A. Bouchenak, I. M. Batiha, M. Aljazzazi, I. H. Jebril, M. Al-Horani, and R. Khalil, "Atomic exact solution for some fractional partial differential equations in Banach spaces," *Partial Differential Equations in Applied Mathematics*, vol. 9, 2024, doi: 10.1016/j.padiff.2024.100626.

[17] I. M. Batiha, I. H. Jebril, A. A. Al-Nana, and S. Alshorm, "A simple harmonic quantum oscillator: fractionalization and solution," *Mathematical Models in Engineering*, vol. 10, pp. 1-8, 2024, doi: 10.21595/mme.2024.23904.

[18] I. M. Batiha, J. Oudetallah, A. Ouannas, A. A. Al-Nana, and I. H. Jebril, "Tuning the fractional-order PID-controller for blood glucose level of diabetic patients," International Journal of Advances in Soft Computing and its Applications, vol. 13, no. 2, pp. 1–10, 2021.

[19] I. M. Batiha, S. A. Njadat, R. M. Batyha, A. Zraiqat, A. Dababneh, and S. Momani, "Design fractional-order PID controllers for single-joint robot arm model," *International Journal of Advances in Soft Computing and its Applications*, vol. 14, no. 2, pp. 96–114, 2022, doi: 10.15849/IJASCA.220720.07.

[20] R. Alika, E. M. Mellouli, and E. H. Tissir, "Adaptive cruise control of the autonomous vehicle based on sliding mode controller using arduino and ultrasonic sensor," *Journal of Robotics and Control*, vol. 5, no. 1, pp. 301–311, 2024, doi: 10.18196/jrc.v5i1.20519.

[21] N. M. Alyazidi, A. M. Hassanine, M. S. Mahmoud, and A. Ma'arif, "Enhanced trajectory tracking of 3D overhead crane using adaptive sliding-

mode control and particle swarm optimization," *Journal of Robotics and Control*, vol. 5, no. 1, pp. 253–262, 2024, doi: 10.18196/jrc.v5i1.18746.

[22] E. Kurniawan, E. B. Pratiwi, H. Adinanta, S. Suryadi, J. A. Prakosa, P. Purwowibowo, S. Wijonarko, T. Maftukhah, D. Rustandi, and M. Mahmudi, "A performance evaluation of repetitive and iterative learning algorithms for periodic tracking control of functional electrical stimulation system," *Journal of Robotics and Control*, vol. 5, no. 1, pp. 205–216, 2024, doi: 10.18196/jrc.v5i1.20705.

[23] M. Afkar, R. Gavagsaz-Ghoachani, M. Phattanasak, S. Pierfederici, and W. Saksiri, "Local-stability analysis of cascaded control for a switching power converter," *Journal of Robotics and Control*, vol. 5, no. 1, pp. 160–172, 2024, doi: 10.18196/jrc.v5i1.20302.

[24] O. Y. Ismael, M. Almaged, and A. I. Abdulla, "Nonlinear model predictive control-based collision avoidance for mobile robot," *Journal of Robotics and Control*, vol. 5, no. 1, pp. 142–151, 2024, doi: 10.18196/jrc.v5i1.20615.

[25] H. Yadavari, V. T. Aghaei, and S. İkizoğlu, "Addressing challenges in dynamic modeling of stewart platform using reinforcement learning-based control approach," *Journal of Robotics and Control*, vol. 5, no. 1, pp. 117–131, 2024, doi: 10.18196/jrc.v5i1.20582.

[26] P. Chotikunnan, R. Chotikunnan, and P. Minyong, "Adaptive parallel iterative learning control with a time-varying sign gain approach empowered by expert system," *Journal of Robotics and Control*, vol. 5, no. 1, pp. 72–81, 2024, doi: 10.18196/jrc.v5i1.20890.

[27] I. A. Hassan, I. A. Abed, and W. A. Al-Hussaibi, "Path planning and trajectory tracking control for two-wheel mobile robot," *Journal of Robotics and Control*, vol. 5, no. 1, pp. 1–15, 2024, doi: 10.18196/jrc.v5i1.20489.

[28] M. Fuad, F. Umam, S. Wahyuni, N. Fahriani, I. Nurwahyudi, M. I. Darwaman, and F. Maulana, "Towards controlling mobile robot using upper human body gesture based on convolutional neural network," *Journal of Robotics and Control*, vol. 4, no. 6, pp. 856–867, 2023, doi: 10.18196/jrc.v4i6.20399.

[29] V. V. Kravchenko, A. A. Efremov, A. A. Zhilenkov, V. N. Kozlov, G. Kristina, P. Anton, P. Mark, and S. Mikhail, "Mathematical model of a robot-spider for group control synthesis: derivation and validation," *Journal of Robotics and Control*, vol. 4, no. 6, pp. 849–855, 2023, doi: 10.18196/jrc.v4i6.20325.

[30] I. G. M. N. Desnanjaya, A. A. S. Pradhana, I. N. T. A. Putra, S. Widiastutik, and I. M. A. Nugraha, "Integrated room monitoring and air conditioning efficiency optimization using ESP-12E based sensors and PID control automation: a comprehensive approach," *Journal of Robotics and Control*, vol. 4, no. 6, pp. 832–839, 2023, doi: 10.18196/jrc.v4i6.18868.

[31] K. Yamtuan, T. Radomngam, and P. Prempraneerach, "Visual servo kinematic control of delta robot using YOLOv5 algorithm," *Journal of Robotics and Control*, vol. 4, no. 6, pp. 818–831, 2023, doi: 10.18196/jrc.v4i6.19102.

[32] M. N. Huynh, H. N. Duong, and V. H. Nguyen, "A passivity-based control combined with sliding mode control for a DC-DC boost power converter," *Journal of Robotics and Control*, vol. 4, no. 6, pp. 780–790, 2023, doi: 10.18196/jrc.v4i6.20071.

[33] E. A. Nugroho, J. D. Setiawan, and M. Munadi, "Handling four DOF robot to move objects based on color and weight using fuzzy logic control," *Journal of Robotics and Control*, vol. 4, no. 6, pp. 769–779, 2023, doi: 10.18196/jrc.v4i6.20087.

[34] M. Almaged, A. Mahmood, and Y. H. S. Alnema, "Design of an integral fuzzy logic controller for a variable-speed wind turbine model," *Journal of Robotics and Control*, vol. 4, no. 6, pp. 762–768, 2023, doi: 10.18196/jrc.v4i6.20194.

[35] I. M. Batiha, Z. Chebana, T. E. Oussaeif, A. Ouannas, and I. H. Jebril, "On a weak Solution of a fractional-order temporal equation," *Mathematics and Statistics*, vol. 10, no. 5, pp. 1116–1120, 2022, doi: 10.13189/ms.2022.100522.

[36] N. Anakira, Z. Chebana, T. E. Oussaeif, I. M. Batiha, and A. Ouannas, "A study of a weak solution of a diffusion problem for a temporal fractional differential equation," *Nonlinear Functional Analysis and Applications*, vol. 27, no. 3, pp. 679–689, 2022, doi: 10.22771/nfaa.2022.27.03.14.

[37] I. M. Batiha, I. Rezzoug, T. E. Oussaeif, A. Ouannas, and I. H. Jebril, "Pollution detection for the singular linear parabolic equation," *Journal of Applied Mathematics and Informatics*, vol. 41, no. 3, pp. 647–656, 2023, doi: 10.14317/jami.2023.647.

[38] I. Lasiecka and R. Triggiani, "Exact controllability of semilinear abstract systems with application to waves and plates boundary control problems," *Proceedings of the 28th IEEE Conference on Decision and Control*, vol. 3, pp. 2291-2294, 1989, doi: 10.1109/CDC.1989.70581.

[39] A. Moumni and J. Salhi, "Exact controllability for a degenerate and singular wave equation with moving boundary," *Numerical Algebra, Control and Optimization*, vol. 13, no. 2, pp. 194–209, 2023, doi: 10.3934/naco.2022001.

[40] M. Abdelli and A. Hafdallah, "Averaged null controllability for some hyperbolic equations depending on a parameter," *Journal of Mathematical Analysis and Applications*, vol. 495, no. 1, p. 124697, 2021, doi: 10.1016/j.jmaa.2020.124697.

[41] C. Fabre, J. P. Puel, and E. Zuazua, "Approximate controllability of the semilinear heat equation," *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, vol. 125, no. 1, pp. 31–61, 1995, doi: 10.1017/S0308210500030742.

[42] N. I. Mahmudov, "Approximate controllability of evolution systems with nonlocal conditions," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 68, no. 3, pp. 536–546, 2008, doi: 10.1016/j.na.2006.11.018.

[43] R. Sakthivel, Y. Ren, and N. I. Mahmudov, "On the approximate controllability of semilinear fractional differential systems," *Computers & Mathematics with Applications*, vol. 62, no. 3, pp. 1451-1459, 2011, doi: 10.1016/j.camwa.2011.04.040.

[44] Y. Cheng, S. Gao, and Y. Wu, "Exact controllability of fractional order evolution equations in Banach spaces," *Advances in Continuous and Discrete Models*, vol. 2018, pp. 1–16, 2018, doi: 10.1186/s13662-018-1794-5.

[45] Z. Fan and G. M. Mophou, "Remarks on the controllability of fractional differential equations," *Optimization*, vol. 63, no. 8, pp. 1205–1217, 2014, doi: 10.1080/02331934.2014.906417.

[46] Z. Chebana, T. E. Oussaeif, A. Ouannas, and I. M. Batiha, "Solvability of Dirichlet problem for a fractional partial differential equation by using energy inequality and Faedo-Galerkin method," *Innovative Journal of Mathematics*, vol. 1, no. 1, pp. 34–44, 2022, doi: 10.55059/ijm.2022.1.1/4.

[47] I. M. Batiha, "Solvability of the solution of superlinear hyperbolic Dirichlet problem," *International Journal of Analysis and Applications*, vol. 20, no. 62, pp. 1–9, 2022, doi: 10.28924/2291-8639-20-2022-62.

[48] I. M. Batiha, Z. Chebana, T. E. Oussaeif, A. Ouannas, S. Alshorm, and A. Zraiqat, "Solvability and dynamics of superlinear reaction diffusion problem with integral condition," *IAENG International Journal of Applied Mathematics*, vol. 53, no. 1, pp. 113–121, 2023.

[49] I. M. Batiha, Z. Chebana, T. E. Oussaeif, A. Ouannas, I. H. Jebril, and M. Shatnawi, "Solvability of nonlinear wave equation with nonlinear integral Neumann conditions," *International Journal of Analysis and Applications*, vol. 21, no. 34, pp. 1–19, 2023, doi: 10.28924/2291-8639-21-2023-34.

[50] I. M. Batiha, A. Ouannas, R. Albadarneh, A. A. Al-Nana, and S. Momani, "Existence and uniqueness of solutions for generalized Sturm–Liouville and Langevin equations via Caputo–Hadamard fractional-order operator," *Engineering Computations*, vol. 39, no. 7, pp. 2581–2603, 2022, doi: 10.1108/EC-07-2021-0393.

[51] T. E. Oussaeif, B. Antara, A. Ouannas, I. M. Batiha, K. M. Saad, H. Jahanshahi, A. M. Aljuaid, and A. A. Aly, "Existence and uniqueness of the solution for an inverse problem of a fractional diffusion equation with integral condition," *Journal of Function Spaces*, vol. 2022, pp. 1–9, 2022, doi: 10.1155/2022/7667370.

[52] I. M. Batiha, N. Barrouk, A. Ouannas, and W. G. Alshanti, "On global existence of the fractional reaction-diffusion system's solution," *International Journal of Analysis and Applications*, vol. 21, no. 11, pp. 1–15, 2023, doi: 10.28924/2291-8639-21-2023-11.

[53] Z. Zhang and J. Wei, "Some results of the degenerate fractional differential system with delay," *Computers & Mathematics with Applications*, vol. 62, no. 3, pp. 1284–1291, 2011, doi: 10.1016/j.camwa.2011.03.061.

[54] M. Plekhanova and G. Baybulatova, "Multi-term fractional degenerate evolution equations and optimal control problems," *Mathematics*, vol. 8, no. 483, pp. 1–9, 2020, doi: 10.3390/math8040483.

[55] R. A. Bandaliyev, I. G. Mamedov, A. B. Abdullayeva, and K. H. Safarova, "Optimal control problem for a degenerate fractional differential equation," *Lobachevskii Journal of Mathematics*, vol. 42, pp. 1239–1247, 2021, doi: 10.1134/S1995080221060056.

[56] V. Vijayakumar, S. F. Aldosary, and K. S. Nisar, "Exact controllability results for Sobolev-type Hilfer fractional neutral delay Volterra-Fredholm integro-differential systems," *Fractal and Fractional*, vol. 6, no. 81, pp. 1–21, 2022, doi: 10.3390/fractalfract6020081.
[57] B. Ahmed, O. Taki-Eddine, R. Imad, and C. Zainouba, "Solvability of solution of singular and degenerate fractional nonlinear parabolic Dirichlet problems," *Bulletin of the Institute of Mathematics*, vol. 17, no. 1, pp. 105–123, 2022, doi: 10.21915/BIMAS.2022104.
[58] B. Antara, O. Taki-Eddine, and R. Imad, "Unique solvability of a Dirichlet problem for a fractional parabolic equation using energy-inequality method," *Methods of Functional Analysis and Topology*, vol. 26, no. 3, pp. 216–226, 2020, doi: 10.31392/MFAT-npu26 3.2020.03.
[59] D. Liu, "Blow-up for a degenerate and singular parabolic equation with nonlocal boundary condition," *Journal of Nonlinear Sciences and Applications*, vol. 9, no. 1, pp. 208–218, 2016, doi: 10.22436/jnsa.009.01.19.
[60] A. E. Bashirov and N. I. Mahmudov, "On concepts of controllability for deterministic and stochastic systems," *SIAM Journal on Control and Optimization*, vol. 37, no. 6, pp. 1808–1821, 1999, doi: 10.1137/S036301299732184X.