EI-FRI: Extended Incircle Fuzzy Rule Interpolation for Multidimensional Antecedents, Multiple Fuzzy Rules, and Extrapolation Using Total Weight Measurement and Shift Ratio

Maen Alzubi [1]*, Mohammad Almseidin [2], Szilveszter Kovacs [3], Jamil Al-Sawwa [4], Mouhammd Alkasassbeh [5]
[1] Department of Robotics and Artificial Intelligence, Jadara University, Irbid, Jordan
[2, 4] Computer Science Department, Tafila Technical University, Tafila, Jordan
[3] Department of Information Technology, University of Miskolc, Miskolc, Hungary
[5] Department of Computer Science, Princess Summaya University, Amman, Jordan
Email: [1] m.alzubi@jadara.edu.jo
*Corresponding Author

***Abstract*—Traditional fuzzy reasoning techniques demand a condensed fuzzy rule base to conclude a result. Still, due to incomplete data or a deficiency of expertise and knowledge, dense rule bases are not always available. Fuzzy interpolation methods have been widely explored to reasonably allow the interpolation of a fuzzy result using the closest current rules. Fuzzy rule interpolation is a type of fuzzy inference system in which conclusions can be obtained even with a few fuzzy rules. This benefit could be used to adapt the FRI to different application areas that suffer from a lack of knowledge. Alzubi et al. [17] offered a novel interpolative method that uses a weighted average based on the center point of the Incircle of the fuzzy sets. Nevertheless, the interpolated observation does not completely define the actual observation that is provided. In our offered extension to this method, a modification weight measure calculation and a shift technique are included to guarantee that the center point of the observation and the interpolated observation are mapped together. This weight measure calculation and shift technique enabled the capability of extrapolation to be conducted implicitly, which is also improves the performance results of the algorithm in the presence of multiple fuzzy rules and multidimensional priors.**

Keywords*—Fuzzy Rule Interpolation; Incircle-FRI; Antecedent Dimensions; Rule Weight Calculation; Rule Shift Ratios.

I. INTRODUCTION

In situations where fuzzy inference methods are used, the number of inputs is large due to the requirement of covering antecedent and consequent fuzzy partitions and also defining all the relationships between the fuzzy sets of antecedents and consequences. It is well known that as the number of inputs (input dimensions) rises, the size of the rule base and complexity will accordingly grow exponentially. To counter this case, the number of redundant rules can be reduced, which will reduce the system's complexity but may also lead to a sparse rule base.

Nevertheless, traditional fuzzy inference methods are suitable for use in complete or near-complete rule-based scenarios, as shown in Fig. 1, which describes the relationship between inputs and outputs via complete rule bases (Rule 1–Rule 9). Thus, there is at least one rule that covers the new observation. Also, they do not work when the observation does not overlap with any current rules, which gives an empty inference, as shown in Fig. 2, which shows the gap between fuzzy sets and rule bases. Thus, there are no rules covering the observation to give the conclusion. Fuzzy rule interpolation, on the other hand, can derive a suitable conclusion by interpolating results from its current adjacent rules.

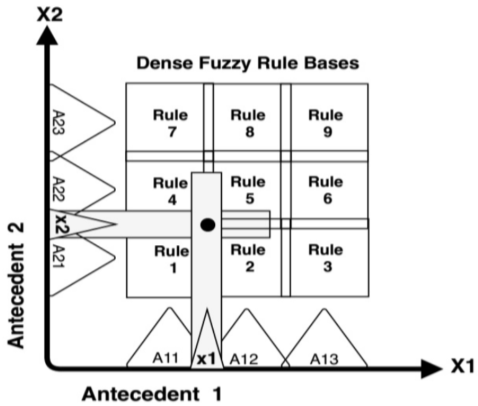

Fig. 1. Dense Fuzzy Rule-Based System

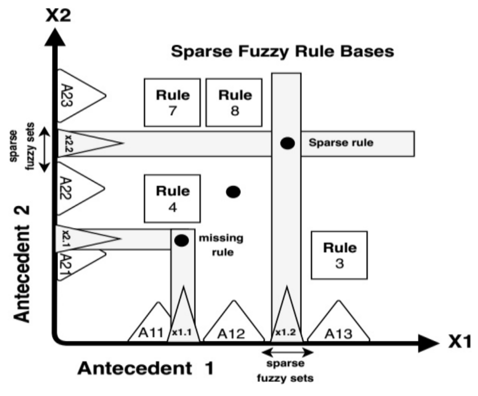

Fig. 2. Sparse Fuzzy Rule-Based System

Fuzzy Rule Interpolation (FRI) techniques can address this issue even when certain observations are not entirely covered by the fuzzy rule system [30]. FRI techniques do not require the use of a complete fuzzy rule base; only the most important fuzzy rules are required to perform the reasoning to get the conclusion, which positively reduces the complexity of fuzzy reasoning. Koczy and Hirota [2] proposed the initial idea of the fuzzy interpolation concept, which is based on linear rule interpolation that uses the α-cuts ($\alpha \in [0,1]$) of the fuzzy sets.

Nevertheless, some FRI methods have been known to produce non-convex interpolated fuzzy sets. In [49], Shi et al. determined the problem and suggested two types of reasoning conditions and methods to produce a "normal and convex" (see [18], [19]) fuzzy set for interpolated conclusions. In [61] proposed a further extension that produces the normal and convex fuzzy sets and handles the problem in [2]. This approach has been proven to ensure the "normality and convexity" of the consequent fuzzy outcome. Ever since the first approach was corrected, many researchers have begun to investigate other methods of interpolation based on different measurements and comparisons. Many researchers have begun to investigate other methods of interpolation based on different measurements and comparisons. In recent years, many methods by different researchers have been proposed and proved [31]-[44], [57], [59], [60]. Many of them were based on the original KH FRI concept, e.g. [2]-[9], [11]-[16].

Successful applications of fuzzy interpolation have been reported in the literature [50]–[54], [56], and [58], including fuzzy control and fuzzy modeling. In particular, fuzzy interpolation has been used for image edge detection [45]. Fuzzy interpolation has also been used to detect slow port scans [46], and FRI has been used to detect IoT-botnet attacks [47]. FRI has been used to enhance the intrusion detection system based on hierarchical bidirectional. Fuzzy interpolation has been applied to support fuzzy modeling in a broad range of application areas, particularly if only limited data are available.

As a special case, in [55], the authors introduced a novel detection method for phishing website attacks while avoiding the issues associated with the deficiencies of knowledge-based representation and binary decision. The proposed detection method was perfumed using the Incircle-FRI method. The proposed method was applied to an open-source benchmark phishing website dataset. The results showed that the accuracy rate of this work is competitive with other methods. which obtained a 97.58% detection rate and effectively reduced the false alerts.

Alzubi et al. [17] introduced a novel approach that uses the weighted computation method to obtain an interpolated fuzzy conclusion. The suggested Incircle-FRI has the following benefits: 1) The convexity and normality of the consequent fuzzy set are ensured; 2) It can work with different membership functions between the antecedent and consequent; 3) It can handle fuzzy interpolative reasoning with logically consistent properties concerning the ratios of fuzziness. However, this method cannot give a desired conclusion in the case of multi-input variables and multiple fuzzy rules, and it also does not perform extrapolation.

In this paper, the extension of the Incircle-FRI method will be introduced to handle the extrapolation and to improve the method with multidimensional rule antecedents, multiple fuzzy rules. We formulate a conversion of the modification of the weight computation and shift ratio process to produce the interpolation of the consequent fuzzy conclusion, which can enable the capability of extrapolation and handle multidimensional rule antecedent variables and multiple fuzzy rules. In the presented method, the modification weight between observation and adjacent rules will be derived according to the general distance instead of the current method of utilizing the utmost distance. This weight calculation is necessary for the performance of the extrapolation capability in the case of single and multiple antecedents. Then a derived fuzzy rule is produced by the shift operation from the fuzzy rules according to the rule distances. The fuzzy conclusion will be computed by having the same shift ratio between the derived rule antecedent and the observation as the shift ratio between the derived rule consequent and the extrapolated consequence.

The rest of the paper is organized as follows: Section (II) reviews the Incircle-FRI method of [17] in general. Section (III) and Section (IV) discuss how the Incircle-FRI method can be applied to multidimensional antecedent variables and the capability of extrapolation using the shifting ratio and weight modification. Section (V) shows the performance of the improved Incircle-FRI approach. The proposed method will be compared to other existing FRI methods using several numerical examples. Finally, the conclusions are discussed in Section (VI).

II. FUZZY RULE INTERPOLATION BASED ON THE INCIRCLE CONCEPT

A. The Incircle-FRI Interpolation method with Single-Antecedent Trigonal Membership Function

The original Incircle-FRI method was created for handling Trigonal Fuzzy-Numbers (TFN) [17]. The TFN was selected for two purposes, its simplicity and popularity. Considering the characteristics of a trigonal-shaped fuzzy set, it can be described by $A = (a_1, a_2, a_3;\ H)$, where the vertices of the Trigonal Fuzzy set on the Cartesian described as $[(a_1, 0), (a_2, H), (a_3, 0)]$, and Height ($H$) is 1, which means the fuzzy set is "normal" as explained in Fig. 3.

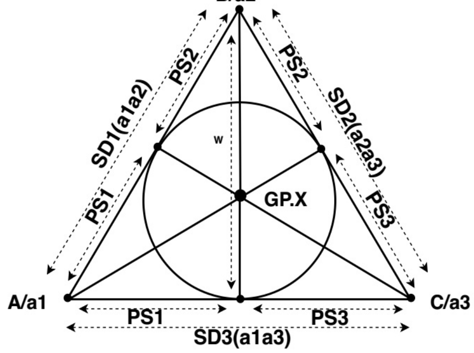

Fig. 3. Trigonal Fuzzy-Number Notations

The proposed fuzzy interpolation is built upon two steps: Step (I) involves determining the key notations of the proposed Incircle-FRI, including the observation and adjacent fuzzy rules. Step (II) involves the determination of the triangular fuzzy conclusion by calculating the center of the fuzzy set referred to by Gergonne Point (GP), and the sides of the trigonal (labeled by (SD_1), (SD_2), and (SD_3). During the calculation, the "sides of fuzziness" of trigonal (labeled by (PS_1), (PS_2), and (PS_3)), which referred to trigonal tangent lengths, (see Fig. 3). Using (1) will be used to determine the Cartesian Coordinate of the Center-Point (GP) of the trigonal fuzzy set (A), as follows using (1):

$$GP_A = (X_A) = (\frac{a_1.\alpha|SD_2| + a_2.\beta|SD_3| + a_3.\gamma|SD_1|}{\alpha|SD_2| + \beta|SD_3| + \gamma|SD_1|} \qquad (1)$$

A fuzzy set (A) is called "normal" if: $\exists x \in U, \mu A(x)$: Height(A) = 1, where Height(A) is the height of fuzzy set A. The sides of fuzziness (PS_1), (PS_2), and (PS_3) of the fuzzy rule antecedents and observation will be specified by using (2).

$$A_{(PS1)} = \frac{(SD_1 + SD_3 - SD_2)}{2} A_{(PS2)} = \frac{(SD_1 + SD_2 - SD_3)}{2} A_{(PS3)} = \frac{(SD_2 + SD_3 - SD_1)}{2} \qquad (2)$$

Secondly: As seen in Fig. 4, suppose that the observation ($A*$) locates amidst the rule antecedent fuzzy sets (A_1) and (A_2). Thus, the conclusion fuzzy set ($B*$) is the consequence obtained through fuzzy interpolation. The conclusion of the Incircle-FRI (interpolated consequent) is computed based on the matching weights of the observation and the two antecedent rules, as well as the conclusion and the two rule consequents with their center points, and the "sides of fuzziness", to produce the trigonal CNF fuzzy conclusions. The procedure of the proposed Incircle-FRI method is introduced in the following steps:

- The adjacent two rules are determined according to the observation.
- The center points and the sides of fuzziness are calculated for the rule antecedents and consequents and for the observation (GP_x, PS_1, PS_3) by (1) and (2).
- The weights between the observation ($A*$) and the adjacent trigonal fuzzy rule antecedents, Rule$_i$ are computed by using (3).

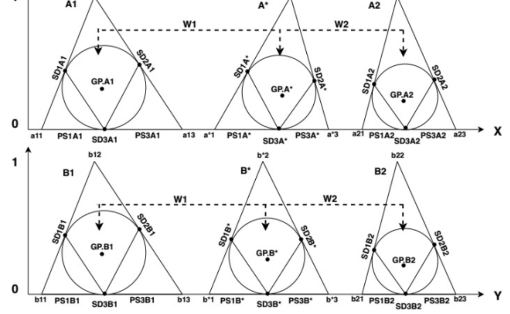

Fig. 4. The Fuzzy interpolation of the trigonal fuzzy sets

$$W_i = 1 - \frac{|GP_x.A^* - GP_x.A_i|}{GP_x.A_2 - GP_x.A_1} \qquad (3)$$

- The conclusion Center-Point (GP_x) is defined by (4):

$$GP_x.B^* = \sum_{i=1}^{2} \quad W_i \times GP_x.B_i \qquad (4)$$

- The sides fuzziness of the conclusion ($B*$), which is based on the neighboring two rules and observation, will be computed by (5):

$$PS_M(B^*) = \left\{PS_M(A^*) \times \sum_{i=1}^{2} \quad W_i \times \frac{PS_M(B_i)}{PS_M(A_i)},\ if\ \exists\ _i PS_M(A_i) > 0\ PS_M(A^*),\ if\ \forall_i PS_M(A_i) = 0\ \right\} \qquad (5)$$

- At the end, the ($B*$) trigonal fuzzy conclusion will be determined by (6):

$$B_1^* = GP_x.B^* - B_{(ps1)}^* B_2^* = GP_x.B^* B_3^* = GP_x.B^* + B_{(ps3)}^* \qquad (6)$$

B. The Incircle-FRI Interpolation Method with Single-Antecedent Trapezoidal Membership Function

It is also possible to apply the Incircle FRI with the trapezoidal fuzzy sets, as a trapezoid can be represented by two triangles, for the left $AL = (a_1, a_2, MP;\ H)$ and for the right $AR = (MP, a_3, a_4;\ H)$. Hence, notations in (3), (4), and (5) will be used to compute trigonal (AL) and trigonal (AR).

Fig. 5 illustrates the center points (GP) of the left and right trapezoidal fuzzy sets, which are defined via ($GP_x.AL$) and ($GP_x.AR$). For the left trigonal (AL) (SDL_1), (SDL_2), (SDL_3), ($PS_1.AL$), and ($PS_3.AL$) will be computed. For the right trigonal (AR) (SDR_1), (SDR_2), (SDR_3), ($PS_1.AR$), and ($PS_3.AR$) are computed. Finally, to derive the conclusion, ($PS_1.AL$) and ($PS_3.AR$) will be used.

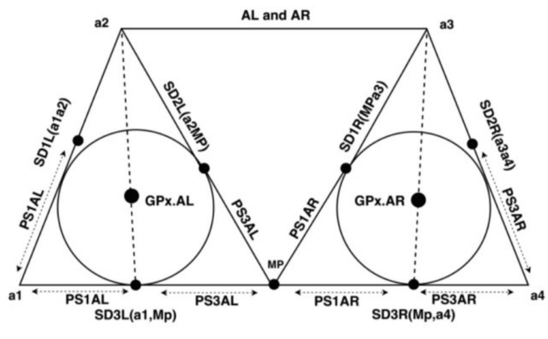

Fig. 5. The trapezoidal fuzzy number notations

The steps to determine the conclusion of the Incircle-FRI for trapezoidal fuzzy sets are the following:

- The reference of the trapezoid will be specified by the average of the two-triangle center-points ($GP_x.AL$) and ($GP_x.AR$) $via\ AVG.GP_x = (GP_x.AL + GP_x.AR)/2$. Then, using (7), the distances will be calculated:

$$D = d(A_i, A^*) = d(AVG.GP_x.A_i, AVG.GP_x.A^*) \qquad (7)$$

- The weights between two adjacent closest fuzzy rules trapezoidal antecedents and the observation can be determined by (8):

$$WL_i = 1 - \frac{|GP_x.A^*L - GP_x.AL_i|}{GP_x.A_2L - GP_x.A_1L} WR_i = 1 - \frac{|GP_x.A^*R - GP_x.AR_i|}{GP_x.A_2R - GP_x.A_1R} \quad (8)$$

(WL_i) and (WR_i) refer to the left and right weights of Rulei, where ($0 \leqslant WL_i \leqslant 1$), ($0 \leqslant WR_i \leqslant 1$), and i = 1, 2. The weight between antecedent and observation fuzzy sets, holding the properties of $(WL_1 + WL_2) = 1$ and $(WR_1 + WR_2) = 1$, as illustrated in Fig. 4.

The Center-Points of the left trigonal ($GP_x.AL$) and the right trigonal ($GP_x.AR$) of the fuzzy conclusion ($B*$) can be defined by (9):

$$GP_x.B^*L = \sum_{i=1}^{2} WL_i \times GP_x.BL_i GP_x.B^*R = \sum_{i=1}^{2} WR_i \times GP_x.BR_i \quad (9)$$

By using (10) and (11) the sides of fuzziness for the ($B*$) fuzzy conclusion could be determined by:

$$PS_M(B^*L) = \left\{ PS_M(A^*L) \times \sum_{i=1}^{2} WL_i \times \frac{PS_M(BL_i)}{PS_M(AL_i)},\ if\ \exists\ {}_i PS_M(AL_i) > 0\ PS_M(A^*L),\ if\ \forall_i PS_M(AL_i) = 0 \right\} \quad (10)$$

$$PS_M(B^*R) = \left\{ PS_M(A^*R) \times \sum_{i=1}^{2} WR_i \times \frac{PS_M(BR_i)}{PS_M(AR_i)},\ if\ \exists\ {}_i PS_M(AR_i) > 0\ PS_M(A^*R),\ if\ \forall_i PS_M(AR_i) = 0 \right\} \quad (11)$$

where $M \in [PS_1, PS_3]$, (PS_1) is the side of left fuzziness from the left trigonal (AL), and (PS_3) is the side of right fuzziness from the right trigonal (AR).

Finally, if the fuzzy set is trapezoidal, the ($B*$) fuzzy conclusion can be specified based on the following four cases:

- If the trigonal fuzzy sets (AL) and (AR) provide identical results, then, the following conclusion may be derived using (12):

$$B^* = [GP_x.B^*L, GP_x.B^*L, GP_x.B^*R, GP_x.B^*R] \quad (12)$$

- If all the values of the left (AL) trigonal are similar, and for the right (AR) trigonal, then, the conclusion values will be defined by (13) :

$$B^* = [GP_{avg}, GP_{avg}, GP_{avg}, GP_{avg}] \quad (13)$$

Where $GP_{avg} = (GP_x B * L + GP_x B * R)/2$.

- If only all the values of the left (AL) trigonal are similar, the conclusion (B_1^*) will be used, in contrast, if only all of the values of the right (AR) trigonal are equal, the conclusion (B_2^*) will be utilized by (14):

$$B_1^* = [GP_x.B^*L,\ GP_x.B^*L,\ GP_x.B^*R,\ GP_x.B^*R + B^*Rps_3]$$
$$B_2^* = [GP_x.B^*L - B^*Lps_1,\ GP_x.B^*L,\ GP_x.B^*R,\ GP_x.B^*R] \quad (14)$$

- If all values of the left (AL) and right (AR) trigonals are dissimilar. The conclusion ($B*$) will be specified by (15):

$$B_1^* = GP_x.B^*L - B^*L_{(PS1)} B_2^* = MP.B^* - GP_x.B^*LB_3^* = MP.B^* - GP_x.B^*RB_4^* = GP_x.B^*R + B^*R_{(PS3)} \quad (15)$$

To calculate the trapezoidal fuzzy set's Mid-Point (MP), the following equation will be used:

$$MP.B* = MP.GP_x B1 + (((MP.GP_x A* - MP.GP_x A1) \times (MP.GP_x B2 - MP.GP_x B1)) / (MP.GP_x A2 - MP.GP_x A1)).$$

The incircle method always generates a convex and normal fuzzy conclusion because of the following condition: $[(B_1^* \leqslant B_2^* \leqslant B_3^* \leqslant B_4^*)]$.

III. THE INCIRCLE-FRI INTERPOLATION METHOD WITH MULTIPLE FUZZY RULES AND MULTIPLE ANTECEDENT UNIVERSES

In this section, fuzzy interpolation with multiple fuzzy rules and multidimensional antecedents will be discussed. The fuzzy rules with multidimensional antecedents have the following format:

R_1:If X_1 is A_{12} and X_2 is A_{12}…and X_m is A_{1m} Then Y is B_1

R_2:If X_1 is A_{21} and X_2 is A_{22}…and X_m is A_{2m} Then Y is B_2

…...………………………………………………………………

R_n:If X_1 is A_{nm} and X_2 is A_{nm}…and X_m is A_{nm} Then Y is B_n

Observation: If X_1 is A_1^* and X_2 is A_2^* … and X_m is A_n^*

Let's assume that there are two antecedent variables, one consequent variable, and two observations ($A*1$), ($A*2$) for the first and second antecedent dimensions, respectively. Each observation is surrounded by four fuzzy rules. For (A_1^*) has two rules (A_{11}) and (A_{21}) on the left and has two rules (A_{31}) and (A_{41}) on the right. For (A_2^*) has two rules (A_{12}) and (A_{22}) on the left, and two rules (A_{32}) and (A_{42}) on the right, which can be represented as follows:

$$(A_{11} \wedge A_{12} \to B_1, A_{21} \wedge A_{22} \to B_2, A_{31} \wedge A_{32} \to B_3,\ and\ A_{41} \wedge A_{42} \to B_4)$$

The ($B*$) fuzzy conclusion, represented by ($b_0^*, b_1^*, b_2^*, b_3^*$), is obtained according to the following five steps:

Step 1: Selecting the closest fuzzy rules to the observation to perform the interpolation ($A_{ij} \preceq A_i^* \preceq A_{ij} + 1$), the fuzzy rules are described as *Rule1, Rule2*, ..., and *Rule$_n$* [20], where (n) denotes the number of the fuzzy rules, as seen by Fig. 6. Having trapezoidal fuzzy sets, the average of Center-Points for the left ($GP_x.AL_{ij}$) and the right ($GP_x.AR_{ij}$) can be computed via: $AVG.GP_x.A = (GP_x.AL_{ij}\ and\ GP_x.AR_{ij})/2)$. Then, to determine the distance between adjacent rules and observations to select the closest fuzzy rules using (16).

$$D = d(A_{ij}, A_i{}^*) = d(AVG.GP_x.A_{ij}, AVG.GP_x.A_i{}^*) \tag{16}$$

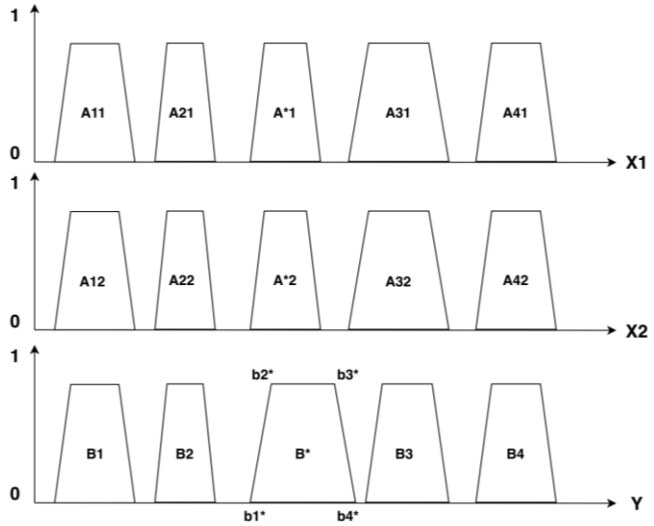

Fig. 6. Four fuzzy rules with two dimensional observations (A∗1, A∗2), the conclusion is (B∗)

Step 2: (17) and (18) will be used to determine the weight between adjacent fuzzy rules and observations, using (19) that will be used to find the Center-Point of the derived conclusion (B').

$$w(s)_{ij} = 1 - \frac{|GP_x.A_j{}^*(s) - GP_x.A(s)_{ij}|}{GP_x.A_{nj}(s) - GP_x.A_{1j}(s)} \tag{17}$$

$$W(s)_i = \frac{\sum_{j=1}^{m} w(s)_{ij}}{\sum_{i=1}^{n} \sum_{j=1}^{m} w(s)_{ij}} \tag{18}$$

$$GP.B'_{(s)} = \sum_{i=1}^{n} W(s)_i GPx.B_{(s)_i} \tag{19}$$

The variable $s \in [L, R]$ indicates the left and right of the weight of the Rulei, holding $(0 \leqslant WL_{ij} \leqslant 1)$, $(0 \leqslant WR_{ij} \leqslant 1)$, and $(i = 1,2,\dots,n)$ indicates the number of the fuzzy rules, $(j = 1,2,.,m)$ refers to the dimension of the antecedent variables. See example on Fig. 6, in case if $n = 4$ and $m = 2$. And $(WL_{1j} + WL_{2j}) = 1$, $(WR_{1j} + WR_{2j} = 1)$, for holding the condition $(W1 + W2) = 1$.

Step 3: The conclusion is Center-Points $(GP_x.B * L)$ and $(GP_x.B * R)$ can be calculated based on (20).

$$\begin{aligned} GP_x.B^*L &= \textstyle\sum_{i=1}^{n} WL_i \times GP_x.BL_i, \\ GP_x.B^*R &= \textstyle\sum_{i=1}^{n} WR_i \times GP_x.BR_i \end{aligned} \tag{20}$$

Step 4: The sides of fuzziness (Left and Right) of the conclusion $(B *)$ can be computed using (21) and (22).

$$PS_M(B_L^*) = \begin{Bmatrix} PS_M(A_L^*) \times \sum_{i=1}^{n} WL_i \times \frac{PS_M(BL_{ij})}{\sum_{j=1}^{m} PS_M(AL_{ij})}, \\ if\ \exists_i\ PS_M(AL_{ij}) > 0\ \frac{\sum_{j=1}^{m} PS_M(AL_j^*)}{m}, \\ is\ \forall_i\ PS_M(AL_{ij}) = 0 \end{Bmatrix} \tag{21}$$

$$PS_M(B_R^*) = \begin{Bmatrix} PS_M(A_R^*) \times \sum_{i=1}^{n} WR_i \times \frac{PS_M(BR_{ij})}{\sum_{j=1}^{m} PS_M(AR_{ij})}, \\ if\ \exists_i\ PS_M(AR_{ij}) > 0\ \frac{\sum_{j=1}^{m} PS_M(AR_j^*)}{m}, \\ is\ \forall_i\ PS_M(AR_{ij}) = 0 \end{Bmatrix} \tag{22}$$

$(M) \in [PS_1, PS_3]$. The fuzzy conclusion $(B *)$ can be computed based on (PS_1) and (PS_3), of antecedents and observation notations. Consequently, if the values of (PS_1) or (PS_3) are greater than 0, the upper portion of (21) and (22) will be used. However, if the values of (PS_1) or (PS_3) are equal to 0, the bottom portion will be applied instead.

Step 5: Finally, using (12), (13), (14), the fuzzy conclusion (B∗) can be determined.

IV. The Extension Of The Incircle-FRI Interpolation Method

The original Incircle FRI method is defined as an "interpolation" method. However, it cannot handle "extrapolation", multiple fuzzy rules, or multidimensional antecedents. This is due to the weight of the derivative and also because of the lack of transformation in the fuzzy range. It is significant to recognize that extrapolation, as outlined in [63], is a "special case" of interpolation. In cases where all the specified nearest rules align on one side of the provided observation, the interpolation method transforms into extrapolation. The process of selecting the closest rules and creating the intermediary rule stays consistent with that used in interpolation.

An essential consideration is the possibility of specific attribute values for the intermediate rule within the specified limitations of the domain space for that attribute. This happens during the construction of the intermediate rule, especially when extrapolation is involved, leading to an intermediate fuzzy rule that extends outside the predefined range.

Additionally, there is the possibility of fuzzified data objects exceeding the confines of the domain space. Consequently, special treatment is deemed necessary for cases of interpolation. For general "interpolation", if either the fuzzified data object or the fuzzy term of the intermediate rule exceeds the input space on a special attribute, such an attribute is skipped when fulfilling the interpolation as this method cannot handle this special case. The following subsections will present the extension and modifications to address these limitations by modifying the derivation of the modification weight calculation and the shifting operation.

A. The Weight Computation Extension

The current weight calculation in (18) is used for two adjacent fuzzy rules but is unsuitable for extrapolation. Fig. 7 represents an example of the various antecedent fuzzy sets with different distances: dis(1) = 4, dis(2) = 7, dis(3) = 10, dis(4) = 3, dis(5) = 6, dis(6) = 36, dis(7) = 40.

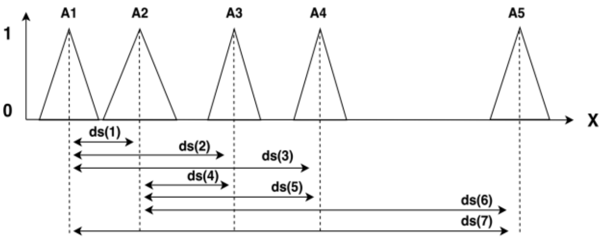

Fig. 7. Fuzzy Sets Distances

To explain that, let fuzzy set (A2) be the observation, and the rest of the fuzzy sets be neighboring rules. Thus, the current implementation will take the distance between fuzzy sets (A1) and (A4), which is dis (3) as the denominator for (18).

The extended weight computation makes the proposed Incircle-FRI method applicable for extrapolation. The weight calculation in (18) works for two adjacent fuzzy rules with a single-dimensional antecedent. To extend the weight calculation to multiple rules and multidimensional antecedents, the weight calculation can be extended to calculate the weights to the closest and farthest rules.

For the extended weight by (23), all the rule antecedent-observation distances are taken into consideration. The denominator is the sum of all the rule antecedent-observation distances. As shown in (24) the weights for each antecedent domain must be normalized. This helps the method preserve the appropriate weight value, i.e., the largest distances have less effect on the weight distribution. This method also preserves the property, if the rule distance is zero, then the corresponding rule weight is equal to one.

$$w_{ij} = 1 - \left| \frac{GP_x A_j^* - GP_x A_{ij}}{\sum_{i=1}^{n} \left| GP_x A_j^* - GP_x A_{ij} \right|} \right| \tag{23}$$

$$W_{ij} = \frac{w_{ij}}{\sum_{i=1}^{n} w_{ij}} \tag{24}$$

where i refers to fuzzy sets, n indicates the number of current fuzzy rules, j refers to the number of antecedent dimensions.

The Incircle-FRI preserves the respective weight values and shows that the distribution of weights stays unaffected by the extreme rules. Table I provides a comparative analysis of weights between the extension Incircle-FRI and the original Incircle-FRI across configurations involving three and four rules. In instances with three rules, the weights pertain to observations of individual rules at a closer distance, whereas in configurations applying four rules, the weights are evaluated when one of the closest rules is slightly further away.

TABLE I. WEIGHT COMPARISON BETWEEN INCIRCLE-FRI AND EXTENSION INCIRCLE-FRI

Distances	Incircle-FRI		Extension Incircle-FRI	
	3 Rules	4 Rules	3 Rules	4 Rules
WA1A2	0.353	0.324	0.346	0.306
WA1A3	0.412	0.333	0.3845	0.313
WA1A4	0.235	0.306	0.2695	0.293
WA1A5	--	0.036	-	0.088

The results derived from the extension Incircle-FRI indicate a high weight allocation for a rule location at the greatest distance from the observation in comparison to the original Incircle-FRI. Specifically, the original method augmented the weight of WA1A4 from 0.235 to 0.306 and assigned a weight of 0.036 to WA1A5 upon the inclusion of the fourth rule. In contrast, our extension of the Incircle-FRI method demonstrates a more acceptable distribution of weights, assigning a weight of 0.088 to WA1A5 for the farthest rule. This highlights that the proposed method not only benefits closer rules but also apportions substantial weight to the rule location at the furthest rule distance.

Let us take an extrapolation scenario shown in Fig. 7, where fuzzy set (A1) serves as the observation and fuzzy sets (A2), (A3), and (A4) represent the three closest rules. The distances between all fuzzy sets remain consistent. In this instance, the distance between the extreme rules is denoted by dis(5). When calculating the weight values for (WA1A3) and (WA1A4) using (18), a negative weight arises due to the numerator exceeding the denominator. To address this issue, (23) presents a solution, which will also assist in establishing the extrapolation capability. Further discussion on this aspect will be expounded upon in subsequent sections.

B. The Shift Ratio Extension

The conclusion can be calculated based on the interpolation of the center points of the two nearest fuzzy rule consequences, applying the corresponding weights calculated by (3) (See e.g. in Fig. 4). However, as shown in Fig. 8, when three rules are considered, the derived observation $(GP_x.A')$ and observation $(A*)$ do not have the same center point. $(GP_x.A')$ can be calculated using a modification weight, where $(WA_1A_2) = 0.34$, $(WA_2A_3) = 0.38$, and $(WA_2A_4) = 0.269$, and with the Center-Points for $(A_1) = 3$, $(A_2) = 8$, $(A_3) = 10$, and $(A) = 5$. The Center-Point of $(GP_x.A_j')$ can be determined by using (25).

$$GPA_j' = \sum_{i=1}^{n} w_{ij} GPA_{ij} \tag{25}$$

where $i = 1,2,...,n$ refers to fuzzy rules, $j = 1,2,...,m$ refers to the number of antecedent dimensions. In the example shown in Fig. 8, $(GP_x.A')$ is calculated, which is (5.648).

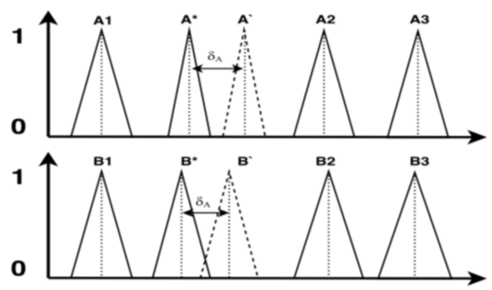

Fig. 8. Interpolation of three rules

Thus, the derived observation $(GP_x.A')$ to the original observation $(GP_x.A*)$ to align the same center point. By using the distance between the center point of $(GP_x.A')$ and $(GP_x.A*)$. using (26) and (27) will be used to calculate the shift ratio.

$$\delta_A = \frac{\sum_{j=1}^{m} \delta_{A_j}}{m}, \tag{26}$$

Where

$$\delta_{A_j} = \frac{GP_x A_j^* - GP_x A_j'}{|GP_x A_{nj} - GP_x A_{1j}|} \tag{27}$$

The A_{nj} is the last fuzzy rule in the antecedent, where $j = 1, 2, \ldots, m$ represents the number of antecedents dimensions.

$(GP_x . A_{nj})$ denotes the center point of the antecedent dimension. Using the distance of the center points between the observation and the derived observation, the distances of the center points for all the antecedent dimensions can be calculated. Therefore, (27) calculates the shift ratio over an antecedent domain. (28) calculates the conclusion as a weighted average with the corresponding rule weights according to the antecedent distances (27).

Based on (δA) and the derived conclusion $(GP_x B')$ by using (19), the conclusion Center-Point $(GP_x . B*)$ can be calculated by (28):

$$GPx.B^* = GPx.B' + \delta_A(|GPx.B_n GPx.B_1|). \tag{28}$$

In the example in Fig. 8, the center-point of the conclusion $(GP_x . B*)$ is (4.615). This leads us to the fact that the weight modification and shift ratio allow the extended Incircle-FRI to work as an extrapolation. Referring to Fig. 8, assuming that, the two fuzzy rules, $(A_2 \Rightarrow B_2)$, $(A_3 \Rightarrow B_3)$, and observation $(A*)$ is not in between the two fuzzy rules (extrapolation). Consequently, $(WA * A_2)$ is (0.60) and $(WA * A_3)$ is (0.40) using (24), $(GP_x . A')$ is (9.6) using (25), by (19), the $(GP_x . B')$ is (8.20). The (δA) is (-1.512) calculated by using (26), $(B_3 - B_2) = 3$ is the difference between the consequent's fuzzy sets, and finally the extrapolated Center-Point of the conclusion $(GP_x B*)$ using (28) is equal (3.66).

Fig. 9. represents a summary diagram of how the Incircle-FRI method works based on the details found in this research. It represents the beginning of the algorithm by determining whether the new observation falls within the current rules or is outside the scope of the current rules. Accordingly, we will determine whether to use the original or modified weights.

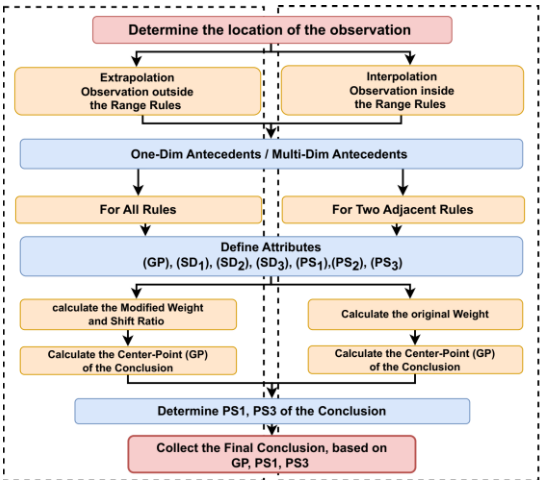

Fig. 9. Interpolation of three rules

V. Experimental Results

In this section, four numerical examples will be introduced to compare the performance of the proposed extended Incircle-FRI with some other FRI methods, which can be found in the literature: KH [2], KH stabilized [21], VKK [22], CCL [23], HS [10], HTY [24], HCL [25], MACI [26], IMUL [27], and CRF [28]. These examples simulate the validity of the modified weight and shifting ratio processes for the extension of the Incircle-FRI method. This is intended to demonstrate the extension of the Incircle-FRI approach.

The four examples will be used to test the extension of the Incircle-FRI method and other FRI methods: The first example will be used to test the validity of FRI methods in the case of a single antecedent with two fuzzy rules. The second example will be used in the case of multi-antecedent variables with three fuzzy rules. The third example will be used to test the validity of FRI methods in the case of extrapolation with a single antecedent part and two fuzzy rules. The last example will be used to test the validity of FRI methods in the case of extrapolation with multiple antecedent variables and two fuzzy rules. performs in the context of multidimensional antecedent variables, multi-rules, and extrapolation properties. To perform these examples, the MATLAB FRI toolbox was used.

Where the FRI toolbox was developed by Z.C. Johanyák et al. [62]. It is implemented in MATLAB and Octave environments [1], [29]. The main goal of the FRI toolbox is to unify different fuzzy interpolation methods, which could be used to evaluate the current FRI methods. The current version of the FRI toolbox is available to download in [48]. It includes the various FRI methods. The package of the FRI toolbox contains software with a graphical user interface, providing easy-to-use access.

A. *Example 1: Testing the FRI Methods with Single-Antecedent Variable*

The purpose of this example is to show how well the Incircle-FRI performs when all of the fuzzy rule antecedents, consequents, and observations are trigonal fuzzy sets. Based on the attributes and results of FRI methods, as shown in Fig. 10 and Table II, we conclude the following facts:

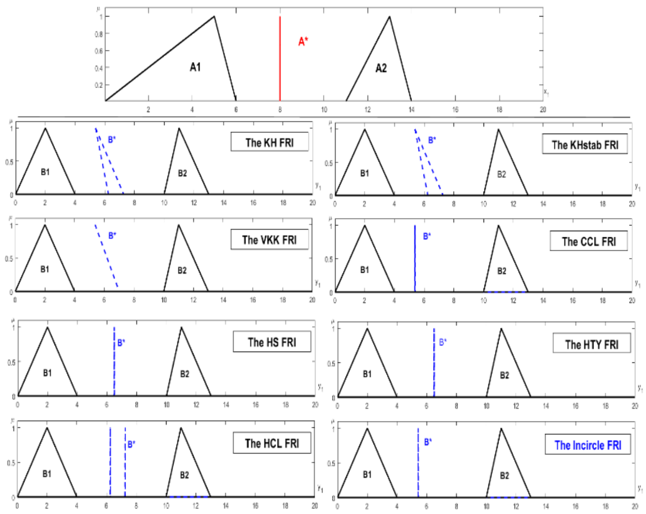

Fig. 10. The fuzzy conclusions of Example 1

The conclusion of Incircle-FRI generated "a singleton type of fuzzy set" using (4), where the center-point of $(GP_x.B*)=5.4$. The left-fuzziness (PS_1) is (0) and the right-fuzziness (PS_3) is (0), which were computed using (5). The conclusion value $(B*)$ of the proposed Incircle-FRI method using (6) is [5.42, 5.42, 5.42].

The proposed Incircle-FRI shows better performance compared to the VKK, HCL, KH, and KHstabilized FRI methods, as they generate non-convex fuzzy conclusions and outperform CCL, HS, HTY, as they have singleton conclusions.

TABLE II. Fuzzy Interpolative Reasoning Results of Example 1 with Triangular Fuzzy Sets of All Fuzzy Rule Bases

Attribute Values	Methods	Results of Fuzzy Interpolative reasoning
A_1=[0 5 6] A_2=[11 13 14] B_1=[0 2 4] B_2=[10 11 13] A^*=[8 8 8]	KH-FRI KHstabilized-FRI VKK-FRI CCL-FRI HS-FRI HTY-FRI HCL-FRI	B*=(7.27 5.38 6.25) B*=(7.27 5.38 6.25) B*=(7.00 5.38 7.00) B*=(5.38 5.38 5.38) B*=(6.49 6.49 6.49) B*=(6.49 6.49 6.49) B*=(7.27 - 6.25)
The Incircle-FRI		B*=(5.42 5.42 5.42)

Note: the sign (-) indicates no clear evidence for the method to handle the case in the example

B. *Example 2: Testing the FRI Methods with Multiple-Antecedent Variables and Multiple Fuzzy Rules*

The purpose of this example is to show how well the Incircle-FRI performs in the case of multiple fuzzy rules and multidimensional antecedents. Supposing that the three fuzzy rules $(A_{11} \wedge A_{12} \Rightarrow B_1)$, $(A_{21} \wedge A_{22} \Rightarrow B_2)$, $(A_{31} \wedge A_{32} \Rightarrow B_3)$, and two-dimensional observations (A_1^*) =[3.5, 5, 5, 7], (A_2^*) =[5, 6, 6, 7]. Thus, based on attribute values and the conclusions of the FRI methods as shown in Fig. 11 and Table III, the following outcomes are derived:

The derived observations of the antecedent variables (A_1') and (A_2') using (25) are $(AL.A_1')$ =5.30, $(AR.A_1')$ = 5.74, $(AL.A_2')$ = 5.30, and $(AR.A_2')$ = 5.74. (26) and (27) will be used to derive the center-points (GP) of the observation and shift ratio calculation of the antecedent variables, thus, (δA_1) = (AL= -0.373 and AR= 0.0373) and (δA_2) = (AL= -0.0373 and AR=0.0373). Based on (26) the average shift ratios are $(\delta A_1) \approx 0$ and $(\delta A_2) \approx 0$.

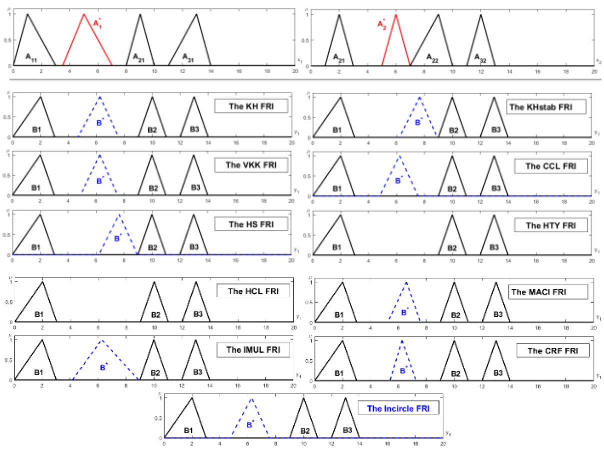

Fig. 11. The fuzzy conclusions of Example 2

TABLE III. Fuzzy Interpolative Reasoning Results of Example 2 with Multiple Fuzzy Rules Having Multiple Antecedents

Attribute Values	Methods	Results of Fuzzy Interpolative reasoning
A_{11}=[0 1 1 3] A_{12}=[1 2 2 3] B_1=[0 2 2 3] A_{21}=[8 9 9 10] A_{22}=[7 9 9 10] B_2=[9 10 10 11] A_{31}=[11 13 13 14] A_{32}=[11 12 12 13] B_3=[12 13 13 14] A^*_1=[3.5 5 5 7] A^*_2=[5 6 6 7]	KH-FRI KHstabilized-FRI VKK-FRI CCL-FRI HS-FRI HTY-FRI HCL-FRI MACI-FRI IMUL-FRI CRF-FRI	B*=(4.67 6.24 6.24 7.57) B*=(6.21 7.66 7.66 8.9) B*=(4.87 6.26 6.26 7.53) B*=(4.79 6.22 6.22 7.54) B*=(6.19 7.65 7.65 8.96) B*=(-) B*=(-) B*=(5.23 6.57 6.57 7.57) B*=(4.1 6.25 6.25 8.89) B*=(5.39 6.25 6.25 7.25)
The Incircle-FRI		B*=(4.73 6.27 6.27 7.55)

Note: the sign (-) indicates no clear evidence for the method to handle the case in the example

The center point of the derived fuzzy conclusion is calculated using (19), where $(GP_x.BL')$ is (6.2) and $(GP_x.BR')$ is (6.2). The center points for the left-trigonal and the right-trigonal were computed by using (28), resulting in $(GP_x.B*L)$ = (6.2) and $(GP_x.B*R)$ = (6.2) respectively, based on the average shift ratios. Because the fuzzy set used is trapezoidal, represented by two trigonal (AL and AR) fuzzy sets, the sides of fuzziness $(AL.PS_1)$ and $(AR.PS_3)$ could be calculated using (21) and (22), therefore, $(AL.PS_1.B*)$= (1.5) and $(AR.PS_3.B*)$ = (1.2). The conclusion $(B*)$ is (4.73, 6.27, 6.27, 7.55).

Accordingly, both the HCL and the HTY methods show no clear evidence of being able to deal with multidimensional antecedent variables. On the other hand, Fig. 11 describes that the convex and normal results were produced by the KH technique, KHstabilized, VKK, the HS, CCL, MACI, IMUL, and CRF, as well as with the suggested Incircle-FRI.

C. *Example 3: Testing the FRI Methods with an Extrapolation Case with a Single-Antecedent Variable*

This example is to show the performance of the Incircle-FRI in the case if the location of the observation $(A*)$ is not in-between the fuzzy rules $((A_1 \Rightarrow B_1)$, $(A_2 \Rightarrow B_2))$ (Extrapolation).

All the fuzzy sets in this example are trigonal fuzzy sets. Fuzzy extrapolation is a problem that arises when we must make predictions outside of the range of observed data. It can occur, e.g., when all the rules are located on the right side of an observation, as shown in Fig. 12. This can lead to difficulties in determining the fuzzy rules for interpolation. Consequently, to solve this issue, the modification weight and shift ratio will be used.

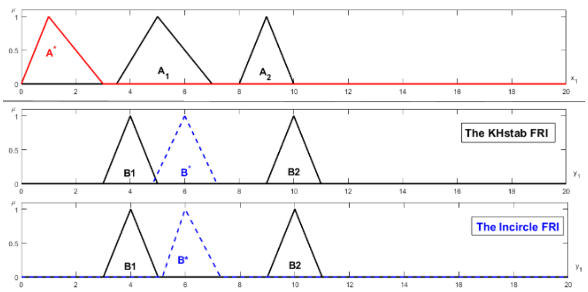

Fig. 12. The fuzzy conclusions of Example 3

Accordingly, the derived observation $(GP_x A')$ can be calculated using (25), which is $(GP_x A')$ = 6.33. By (26) the shift ratio is (δA) = -1.32. Using (19) the derived conclusion $(GP_x B')$ is calculated, where $(GP_x B')$ = 5.99, and the center-point (GP) of $(GP_x B*)$ = 5.99, which was computed by (28). The left fuzziness-side and right fuzziness-side according to (5) are $(PS_1 B*)$ = 0.83 and $(PS_3 B*)$ = 1.28, respectively. According to (6), the extrapolated fuzzy conclusion is $(B*)$ = (5.16, 5.96, 7.28).

Based on Table IV, which explains the numerical results of this example, there is no conclusion for the KH, VKK, HCL, HTY, MACI, IMUL, and CRF methods to address the fuzzy interpolation with extrapolation property. Fig. 11 explains the properties of the convex and normal conclusions for the results of the FRI methods, which were produced only by the KH stabilized method as well as by the proposed Incircle-FRI method.

TABLE IV. FUZZY INTERPOLATIVE REASONING RESULTS OF EXAMPLE 3 WITH TRIANGULAR FUZZY SETS IN CASE EXTRAPOLATION OBSERVATION

Attribute Values	Methods	Results of Fuzzy Interpolative reasoning
A_1=[3.5 5 7]	KH-FRI	B*=(-)
A_2=[8 9 10]	KHstabilized-FRI	B*=(4.82 6 6 7.18)
B_1=[3 4 5	VKK-FRI	B*=(-)
B_2=[9 10 11	HTY-FRI	B*=(-)
A^*=[0 1 3]	HCL-FRI	B*=(-)
	MACI-FRI	B*=(-)
	IMUL-FRI	B*=(-)
	CRF-FRI	B*=(-)
The Incircle-FRI		B*=(5.16 5.99 7.28)

Note: the sign (-) indicates no clear evidence for the method to handle the case in the example

D. *Example 4: Testing the FRI Methods with an Extrapolation Case with a Multiple-Antecedent Variables and Multiple Fuzzy Rules*

This example is to show the performance of the Incircle-FRI with multidimensional antecedents as shown in Fig. 13, where the locations of the observations are not in-between the fuzzy rules. All the fuzzy sets of this example are trigonal fuzzy sets. Consequently, to solve this issue, the modification weight and shift ratio will be used with the same steps used for interpolation. Table V shows the results obtained by KHstabilized, HS, CCL, and Incircle-FRI methods. The derived observations $(GP_x A_1')$ and $(GP_x A_2')$ can be calculated using (25), which are $(GP_x A_1')$ = 10.32 and $(GP_x A_2')$ = 9.97.

TABLE V. FUZZY INTERPOLATIVE REASONING RESULTS OF EXAMPLE 4 WITH TRIANGULAR FUZZY SETS IN CASE EXTRAPOLATION OBSERVATION

Attribute Values	Methods	Results of Fuzzy Interpolative Reasoning
A_{11}=[8 9 10]	KHstabilized-FRI	B*=(10 11 11.9)
A_{21}=[11 13 14]	HS-FRI	B*=(6.3 7.7 8.7)
	CCL-FRI	B*=(6.1 7.01 8.5)
A_{12}=[7 9 10]	KH-FRI	B*=(-)
A_{22}=[11 12 13]	VKK-FRI	B*=(-)
	HTY-FRI	B*=(-)
B_1=[9 10 11]	HCL-FRI	B*=(-)
B_2=[12 13 14	MACI-FRI	B*=(-)
A^*_1=[3.5 5 7]	IMUL-FRI	B*=(-)
A^*_2=[5 6 7]	CRF-FRI	B*=(-)
The Incircle-FRI		B*=(6.1 7 8.4)

Note: the sign (-) indicates no clear evidence for the method to handle the case in the example

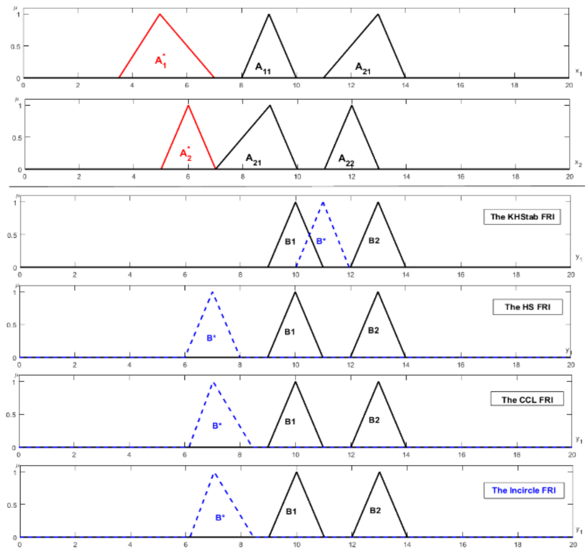

Fig. 13. The fuzzy conclusions of Example 4

Using the derived observations can calculate the average of the shift ratio with (26), which is (δA) = -1.32. The center-point of the derived consequent is specified by (19), giving us $(GP_x B')$= 10.9, and the Center-Point (GP) of $(GP_x B*)$ = 7.02, which was computed by (28). The left fuzziness-side and right fuzziness-side according to (5) are $(PS_1 B*)$ = 0.87 and $(PS_3 B*)$ = 1.42, respectively. According to (6), the extrapolated fuzzy conclusion is $(B*)$ = (6.1, 7.0, 8.4).

According to FRI methods results, which are shown in Table V and Fig. 13, there is no obvious indication for the KH method, VKK-FRI, HCL-FRI, HTY-FRI, MACI-FRI, IMUL-FRI, and CRF-FRI to handle extrapolation case with multi-antecedent variables. In contrast, the KHstabilized-FRI, HS-FRI, CCL-FRI, and the Incircle-FRI generate a conclusion with convex and normal results.

Table VI presents a summary of the outcomes obtained from the chosen Fuzzy Rule Interpolation (FRI) methods based on the preceding examples. In this representation, the addition symbol (√) signifies that the technique achieved results consistent with the example's characteristics. whereas the minus sign (×) denotes instances where the method did not achieve results consistent with the example's characteristics.

TABLE VI. SUMMARY OF THE FRI METHODS AND THEIR CONFORMITY TO EXAMPLES (1, 2, 3, 4)

Methods	**Example (1)**	**Example (2)**	**Example (3)**	**Example (4)**
KH-FRI	√	√	×	×
KHstb-FRI	√	√	√	√
VKK-FRI	√	√	×	×
CCL-FRI	√	√	√	√
HS-FRI	√	√	√	√
HTY-FRI	√	×	×	×
HCL-FRI	√	×	×	×
MACI-FRI	√	√	×	×
CRF-FRI	√	√	×	×
IMUL-FRI	×	√	×	×
Incircle-FRI	√	√	√	√

VI. CONCLUSIONS

The extension of the Incircle-FRI method includes enhancements to adapt to multiple fuzzy rules and multiple antecedent dimensions, as well as extrapolation capabilities. This extension concerns a modified computation for both weight and shift ratio.

Wherein the modified weight considers all distances between the observation and rule antecedents, differing from the original Incircle-FRI, which exclusively considers the distance to the closest rule antecedents. Additionally, an extended shift ratio is introduced between the derived rule and observation to determine the fuzzy conclusion.

The proposed modifications in weight and shift ratio calculation exhibit proficiency in ascertaining both interpolated and extrapolated fuzzy conclusions. The comparative analysis and results from the provided examples demonstrate the superior performance of the proposed extended Incircle-FRI method across all scenarios.

This approach provides convexity and normality fuzzy inferences for multiple fuzzy rules characterized by multidimensional antecedents, including extrapolation scenarios. The applicability of Incircle-FRI depends on the inherent characteristics of the data and the specific problem domain under consideration. Also, particularly well-suited for scenarios characterized by sparse rule bases, Incircle-FRI addresses challenges that traditional fuzzy systems encounter in providing precise inferences. In the future, we seek to apply this Incircle-FRI in different areas to determine its effectiveness and its ability to give the desired results, especially in those areas that have insufficient data.

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