On New Results of Stability and Synchronization in Finite-Time for Fitiz-Nagamo Model Using Grownal Inequality and Lyapunov Function

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***Abstract*—Ionic diffusion across cytomembranes plays a critical role in both biological and chemical systems. This paper reexamines the FitzHugh-Nagumo reaction-diffusion system, specifically incorporating the influence of diffusion on the system's dynamics. We focus on the system's finite-time stability, demonstrating that it achieves and maintains equilibrium within a specified time interval. Unlike asymptotic stability, which ensures long-term convergence, finite-time stability guarantees rapid convergence to equilibrium, a crucial feature for real-time control applications. We prove that the equilibrium point of the FitzHugh-Nagumo system exhibits finite-time stability under certain conditions. In particular, we provide a criterion for finite-time stability and derive results using new lemmas and a theorem to guide the system's design for reliable performance. Additionally, the paper discusses finite-time synchronization in reaction-diffusion systems, emphasizing its importance for achieving coherent dynamics across distributed components within a finite time. This approach has significant implications for fields requiring precise control and synchronization, such as sensor networks and autonomous systems. Practical simulations are presented to elucidate the theoretical principles discussed earlier, using the finite difference method (FDM) implemented in MATLAB.**

Keywords*—Fitzhugh-Nagumo Reaction-Diffusion System; Finite-Time Stability; Real-Time Control; Finite-Time Synchronization; Reaction-Diffusion Systems; Finite Difference Method (FDM)

I. Introduction

Neurons are fundamental units in neural systems, pivotal to neuroscience, brain science, and medical technology. While the Hodgkin-Huxley (HH) model [1]–[4] is a classic framework for simulating neural network dynamics, its complexity often necessitates the use of simplified models like the FitzHugh-Nagumo (FHN) model [5], [6]. The FHN model captures key features of neuronal excitability through cubic nonlinearity, providing insights into nerve impulse propagation, separatrix loops, equilibrium bifurcation, and limit cycles [7], [8]. Reaction-diffusion systems (RDSs), described by partial differential equations, are crucial for modeling phenomena such as chemical pattern formation, biological processes, and disease spread [9]–[11]. When extended to include spatial diffusion, the FitzHugh-Nagumo model becomes the FitzHugh-Nagumo reaction-diffusion system (FHN-RDS). This extension allows for an exploration of how neuronal activity propagates spatially, offering insights into neural pattern formation and interactions across spatial domains.

Recent research has highlighted the significance of finite-time stability for accelerating convergence and enhancing system resilience. Works such as [12], [13] have established criteria for finite-time stability in systems with time-varying delays using the Lyapunov-Razumikhin technique, while [14], [15] extended the generalized Gronwall inequality to systems with delays and disturbances. These advancements underscore the growing interest in finite-time stability as a means to achieve rapid convergence in dynamic systems.

Synchronization in dynamical systems involves aligning multiple systems to oscillate coherently and has broad applications [16]–[21]. Approaches to achieve synchronization include linear and nonlinear control techniques [22]–[26], categorized into finite-time synchronization, phase synchronization, and generalized synchronization [27]–[33] . Finite-time synchronization, in particular, ensures systems achieve synchrony within a predetermined time frame, which is crucial for applications requiring precise coordination, such as networked and real-time control systems [34]–[37]. Unlike infinite-time synchronization, which only guarantees eventual convergence, finite-time synchronization ensures convergence within a specific, often short, period [38]–[46]. This feature is essential for performance and effectiveness in scenarios where timely coordination is critical [47]–[51].

Despite these advancements, achieving finite-time synchronization in spatially extended systems like reaction-diffusion systems presents unique challenges [52], [53]. The complexity of spatial and temporal dynamics, nonlinearities, and diffusion-driven interactions complicates synchronization efforts [54]. Current methods often address infinite-time synchronization or simpler systems, leaving a gap in understanding how to synchronize complex systems within finite time [55]–[59]. This gap highlights the need for innovative techniques specifically tailored to the complexities of reaction-diffusion systems [60].

In response to these challenges, this manuscript makes the following contributions:

- **Novel Stability Lemma:** We develop a new lemma specifically for finite-time stability in reaction-diffusion systems. This lemma introduces precise criteria for achieving convergence to equilibrium points within a finite time frame, addressing the limitations of existing stability criteria in capturing the dynamics of FitzHugh-Nagumo reaction-diffusion systems (FHN-RDS).
- **State-Dependent Linear Controllers:** We design and implement state-dependent linear controllers to facilitate rapid synchronization between primary and response systems. The linear control techniques are chosen for their effectiveness in managing the linearizable aspects of FHN-RDS, aiming for efficient and precise synchronization.
- **Extension of Generalized Gronwall Inequality:** We apply and extend the generalized Gronwall inequality to derive sufficient conditions for finite-time stability and synchronization in reaction-diffusion systems. This extension addresses the spatial and temporal complexities inherent in FHN-RDS.
- **Validation through Numerical Simulations:** We conduct comprehensive numerical simulations using the finite difference method (FDM) to test and validate the theoretical results. These simulations provide practical insights and demonstrate the effectiveness of the proposed techniques under realistic conditions.

The paper is organized as follows: Section II explores the FitzHugh-Nagumo reaction-diffusion system, analyzing the impact of ionic diffusion on its dynamics. Section III presents the application of the proposed stability lemma and state-dependent linear controllers for finite-time stability and synchronization in coupled FHN systems. Section IV provides a comprehensive theoretical framework and practical methodology for finite-time synchronization in reaction-diffusion systems, linking theoretical results to real-world applications. Finally, Section V focuses on numerical simulations using FDM, validating the theoretical findings and assessing the performance of the proposed techniques.

II. Model Description

The FitzHugh-Nagumo reaction-diffusion system is widely recognized for modeling the propagation of nerve impulses. Ionic diffusion, which occurs as ions traverse the cytomembrane, plays a critical role in this process. Therefore, it is crucial to incorporate diffusion effects into the model to capture the dynamics accurately. In this manuscript, we reexamine the FitzHugh-Nagumo reaction-diffusion system as described in [24]

$$\begin{cases} \dfrac{\partial U_1(r,z)}{\partial z} = n_1 \Delta U_1 + (a - U_1)(U_1 - 1)U_1 - U_2, r \in \Omega, \\ \dfrac{\partial U_2(r,z)}{\partial z} = n_2 \Delta U_2 + c(bU_1 - U_2), r \in \Omega, z > 0, \\ \dfrac{\partial U_1}{\partial \eta} = \dfrac{\partial U_2}{\partial \eta} = 0, r \in \partial\Omega, z > 0, \\ U_1(r,0) = U_{1,0}(r) > 0, U_2(r,0) = U_{2,0}(r) > 0, \quad r \in \Omega, \end{cases} \tag{1}$$

In this system, Ω represents a bounded domain within $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, and Δ denotes the Laplace operator acting on Ω. The Laplace operator, denoted by Δ, describes the spatial diffusion of the reactants in the system, which is crucial for capturing the spatial propagation of signals in excitable media. The choice of Neumann boundary conditions, $\dfrac{\partial U_1}{\partial \eta} = \dfrac{\partial U_2}{\partial \eta} = 0$, reflects the assumption that there is no flux across the boundary of the domain, a common scenario in biological systems where the boundary represents impermeable membranes or the edges of the tissue [61], [62].

The initial conditions, $U_1(r,0)$ and $U_2(r,0)$, are chosen to be positive functions over the domain Ω. These conditions are critical as they represent the initial state of the membrane potential and recovery variable, respectively. The positivity of the initial values ensures the physiological relevance of the model, as negative values would not be meaningful in the context of nerve impulse propagation.

The parameters n_1 and n_2 represent the diffusion coefficients for the variables U_1 and U_2, respectively. These coefficients are crucial in determining the rate at which the signals spread through the domain. The excitatory threshold a, the parameter b affecting the rest state, and the excitability parameter c are all chosen based on their roles in shaping the dynamics of the system, particularly in determining the conditions under which the system exhibits excitable behavior.

These assumptions and choices are standard in the study of reaction-diffusion systems and are motivated by the need to model the key features of excitable media accurately. The selected boundary and initial conditions, along with the parameter choices, ensure that the model is both mathematically tractable and physiologically relevant. Moreover, they allow for the derivation of analytical results, such as the finite-time stability of equilibrium points, as discussed in subsequent sections.

III. FINITE TIME STABILITY RESULT

Finite-time stability concerns achieving and maintaining system equilibrium within a specified time interval, regardless of initial conditions. Unlike asymptotic stability, which involves approaching equilibrium over an infinite time span, finite-time stability ensures convergence within a predetermined time frame. This property is particularly valuable in real-time control systems and critical applications where rapid stabilization is essential. In this section, we demonstrate that the FitzHugh-Nagumo reaction-diffusion system (1) exhibits finite-time stability at the equilibrium point [63]–[65].

Definition 1 ensures finite-time stability by maintaining the system's proximity to the equilibrium within a specified time frame. Lemma 1 guarantees that solutions are unique and bounded, ensuring predictable system behavior. Lemma 2 relates a function's integral to its gradient, facilitating stability analysis. Lemma 3 uses exponential bounds to regulate function growth, thereby supporting stability. Theorem 1 provides a criterion for assessing the finite-time stability of the equilibrium point, guiding the design and control of the system for reliable performance. First, we determine the equilibrium points of the system (2)

$$\begin{cases} n_1\Delta U_1^* + (a - U_1^*)(U_1^* - 1)U_1^* - U_2^* = 0, \\ n_2\Delta U_2^* + c(bU_1^* - U_2^*) = 0. \end{cases} \tag{2}$$

Herein, we have many equilibrium points depending on the sign of

$$\chi = (a-1)^2 - 4b, \tag{3}$$

- If $\chi > 0$, there are three equilibrium points:

$$(U_1^*, U_2^*) = \left\{(0,0), \left(\frac{a+1\mp\sqrt{\chi}}{2}, \frac{b\left(a+1\mp\sqrt{\chi}\right)}{2}\right)\right\} \tag{4}$$

- If $\chi = 0$, there are two equilibrium points:

$$(U_1^*, U_2^*) = \left\{(0,0), \left(\frac{a+1}{2}, \frac{b(a+1)}{2}\right)\right\}, \tag{5}$$

- If $\chi < 0$, there is a unique equilibrium point:

$$(U_1^*, U_2^*) = (0,0). \tag{6}$$

Definition 1. *[66] The equilibrium point (U_1^*, U_2^*) of system (1) with initial conditions is said to be finite time stable with respect to $\{\delta, \varepsilon, \mathtt{J}\}, \delta < \varepsilon$, if and only if*

$$\|(U_{1,0} - U_1^*, U_{2,0} - U_2^*)\| := \|U_{1,0} - U_1^*\| + \|U_{2,0} - U_2^*\| < \delta, \tag{7}$$

implies

$$\|(U_1(z) - U_1^*, U_2(z) - U_2^*)\| := \|U_1(z) - U_1^*\| + \|U_2(z) - U_2^*\| < \varepsilon, \forall z \in \mathtt{J} = [0, z^*], \tag{8}$$

where

$$\|U_0 - U^*\|^2 = \int_\Omega |U_0(r) - U^*|^2\, dr,$$

and

$$\|U(z) - U^*\|^2 = \int_\Omega |U(r,z) - U^*|^2\, dr.$$

Lemma 1. *[24] Model (1) possesses a globally unique solution (U_1, U_2), and there exists a constant $Q \in \mathbb{R}^+$ such that*

$$U_1(r,z), U_2(r,z) \le Q, \tag{9}$$

for all $(r,z) \in \Omega \times [0, +\infty)$.

Lemma 2. *[57], [58], [67] Let $\Omega \subset \mathbb{R}^m$ be a bounded domain with smooth boundary $\partial\Omega$ of class $\mathcal{C}^2$, $U(r) \in H_0^1(\Omega)$ is a real-valued function and $\left.\frac{\partial U(r)}{\partial \eta}\right|_{\partial\Omega} = 0$. Then*

$$\Upsilon_1 \int_\Omega |U(r)|^2\, dr \le \int_\Omega |\nabla U(r)|^2\, dr, \tag{10}$$

where Υ_1 is defined by the positive eigenvalue of the problem

$$\begin{cases} -\Delta U(r) = \Upsilon U(\mathfrak{r}), & r \in \Omega, \\ \dfrac{\partial U(r)}{\partial \eta} = 0, & r \in \partial\Omega. \end{cases} \tag{11}$$

Lemma 3. *[66] Assume that function $U(z)$ satisfies*

$$U(z) \le F_1(z) + \int_0^z U(s)F_2(s)\, ds, \quad z \in [0, z^*], z^* < \infty, \tag{12}$$

where

$$U(z), F_1(z), F_2(z) \in C([0, z^*]), \quad z^* < \infty,$$

and

$$F_2(z) > 0.$$

If $F_1(z)$ is non decreasing. Then

$$U(z) < F_1(z)\, e^{\int_0^z F_2(s)ds}. \tag{13}$$

Theorem 1. *The equilibrium point* (U_1^*, U_2^*) *of system* (1) *exhibits finite-time stability if and only if the following condition is satisfied:*

$$M = \max\left\{3Q^2 + 2(a+1)Q + a + \frac{cb+1}{2} - n_1\Upsilon_1, \frac{cb+1}{2} - c - n_2\Upsilon_2\right\} > 0. \quad (14)$$

Here, the finite-time stability parameter z^* is defined as

$$z_1^* = \frac{2}{M}\ln\left(\frac{\varepsilon}{\sqrt{2}\delta}\right). \quad (15)$$

Proof. *We describe the formula* (16) *as follows:*

$$\begin{cases} R_1(r,z) = U_1(r,z) - U_1^*, \\ R_2(r,z) = U_2(r,z) - U_2^*. \end{cases} \quad (16)$$

By applying Green's formula and Lemma 2, we obtain

$$\begin{aligned} \frac{\partial}{\partial z}\int_\Omega R_1^2\,dr &= n_1\int_\Omega R_1\Delta R_1\,dr \\ &\quad - \int_\Omega \Bigg[U_1^2 + U_1^{*2} + U_1U_1^* \\ &\quad - (a+1)\left(U_1 + U_1^* - \frac{a}{a+1}\right)\Bigg] R_1^2\,dr - \int_\Omega R_1R_2\,dr \\ &\le -n_1\int_\Omega |\nabla R_1|^2\,dr + \int_\Omega \Big[|U_1|^2 + |U_1^*|^2 + |U_1||U_1^*| \\ &\quad + (a+1)\left(|U_1| + |U_1^*| + \frac{a}{a+1}\right)\Big] R_1^2\,dr - \int_\Omega R_1R_2\,dr \\ &\le \left(3Q^2 + 2(a+1)Q + a - n_1\Upsilon_1\right)\int_\Omega R_1^2\,dr \\ &\quad + \int_\Omega |R_1||R_2|\,dr. \end{aligned} \quad (17)$$

Similarly, we have

$$\begin{aligned} &\frac{\partial}{\partial z}\int_\Omega R_2^2\,dr \\ &\quad = n_2\int_\Omega R_2\Delta R_2\,dr + cb\int_\Omega R_1R_2\,dr - c\int_\Omega R_2^2\,dr \\ &\le -n_2\int_\Omega |\nabla R_2|^2\,dr + cb\int_\Omega R_1R_2\,dr - c\int_\Omega R_2^2\,dr \\ &\le -(n_2\Upsilon_2 + c)\int_\Omega R_2^2\,dr + cb\int_\Omega |R_1||R_2|\,dr. \end{aligned} \quad (18)$$

Therefore, we have

$$\begin{aligned} \int_\Omega R_1^2 + R_2^2\,dr &\le \int_\Omega R_{1,0}^2 + R_{2,0}^2\,dr + \big(3Q^2 + 2(a+1)Q \\ &\quad + a - n_1\Upsilon_1\big)\int_0^z\int_\Omega R_1^2 dr ds - (n_2\Upsilon_2 + c)\int_0^z\int_\Omega R_2^2 dr ds \\ &\quad + (cb+1)\int_0^z\int_\Omega |R_1||R_2|\,dr\,ds, \end{aligned} \quad (19)$$

which implies

$$\begin{aligned} \int_\Omega R_1^2 + R_2^2\,dr &\le \int_\Omega |R_{1,0}|^2 + |R_{2,0}|^2\,dr \\ &+ \left(3Q^2 + 2(a+1)Q + a + \frac{cb+1}{2} - n_1\Upsilon_1\right)\int_0^z \\ &\int_\Omega |R_1|^2\,dr\,ds + \left(\frac{cb+1}{2} - c - n_2\Upsilon_2\right)\int_0^z\int_\Omega |R_2|^2\,dr\,ds. \end{aligned} \quad (20)$$

Consequently, we obtain

$$\begin{aligned} \int_\Omega R_1^2 + R_2^2\,dr &\le \int_\Omega |R_{1,0}|^2 + |R_{2,0}|^2\,dr \\ &+ \max\Bigg\{3Q^2 + 2(a+1)Q + a + \frac{cb+1}{2} - n_1\Upsilon_1, \\ &\frac{cb+1}{2} - c - n_2\Upsilon_2\Bigg\}\int_0^z\int_\Omega R_1^2 + R_2^2\,dr\,ds \\ &\le \delta^2 + \max\Bigg\{3Q^2 + 2(a+1)Q + a + \frac{cb+1}{2} - n_1\Upsilon_1, \\ &\frac{cb+1}{2} - c - n_2\Upsilon_2\Bigg\}\int_0^z\int_\Omega R_1^2 + R_2^2\,dr\,ds. \end{aligned} \quad (21)$$

By assuming $M = \max\left\{3Q^2 + 2(a+1)Q + a + \frac{cb+1}{2} - n_1\Upsilon_1, \frac{cb+1}{2} - c - n_2\Upsilon_2\right\} > 0$ *and considering Lemma 3, we get*

$$\int_\Omega |R_1|^2 + |R_2|^2\,dr \le \delta^2 e^{Mz}. \quad (22)$$

Further, we obtain

$$\|R_1(z)\| + \|R_2(z)\| \le F(z) = \sqrt{2}\delta e^{\frac{M}{2}z}. \quad (23)$$

Thus, the settling time is given by

$$z_1^* = \frac{2}{M}\ln\left(\frac{\varepsilon}{\sqrt{2}\delta}\right). \quad (24)$$

Therefore, system (1) *is stable over a finite duration if* z *is greater than or equal to* z^*.

IV. Finite-Time Synchronization Scheme

Finite-time synchronization in reaction-diffusion systems aims to rapidly achieve coherent dynamics among spatially distributed components within a defined period. This synchronization is crucial for establishing uniform behavior across the system's spatial dimensions. By linking the primary system (1) with the response system (25), synchronization is facilitated [68]–[71].

Achieving finite-time stability and synchronization is essential for ensuring reliable and efficient operation across various applications. When synchronization occurs within a finite time, distributed components converge to a unified state, which is critical for applications such as sensor networks and industrial automation, where timely synchronization is of the essence [72].

For safety-critical systems like autonomous vehicles, predictable stabilization within a finite time frame is key to ensuring consistent and safe operation. Additionally, streamlined control strategies enhance cost-efficiency by reducing implementation costs and operational complexity. This benefits large-scale processes by minimizing reliance on extensive hardware and manual intervention [73], [74].

Minimizing synchronization errors between the primary and response systems improves accuracy and reliability, making this approach valuable for precision-driven fields such as robotics and aerospace. We define the response system (25) as follows:

$$\begin{cases} \dfrac{\partial V_1(r,z)}{\partial z} = n_1 \Delta V_1 + (a - V_1)(V_1 - 1)V_1 - V_2 \\ \qquad\qquad +C,\ r \in \Omega,\ z > 0, \\ \dfrac{\partial V_2(r,z)}{\partial z} = n_2 \Delta V_2 + c(bV_1 - V_2),\ r \in \Omega,\ z > 0, \\ \dfrac{\partial V_1}{\partial \eta} = \dfrac{\partial V_2}{\partial \eta} = 0,\ r \in \partial\Omega, \\ V_1(r,0) = V_{1,0}(r) > 0, \quad V_2(r,0) = V_{2,0}(r) > 0,\ r \in \Omega \end{cases} \tag{25}$$

The objective of this research is to develop a streamlined controller, C, aimed at reducing costs and simplifying implementation. This controller C is crucial for ensuring finite-time synchronization within the system. It is designed to adjust the error dynamics effectively, minimizing the synchronization error e until it reaches zero. This careful tuning is essential not only for maintaining stability but also for reducing implementation costs and complexity.

The ability of the controller C to achieve finite-time synchronization is particularly critical for real-time applications, where systems must quickly reach a coordinated state. By analyzing the error dynamics, the conditions necessary for synchronization can be derived, providing valuable insights into the stability and behavior of the coupled systems. Additionally, the controller's streamlined design ensures it is cost-effective and easy to implement, which is especially important in practical scenarios with limited resources and computational power. The effectiveness of the controller and the synchronization analysis can be validated through both theoretical derivations and numerical simulations, offering robust evidence of its capability to achieve synchronization within the desired time frame [75]–[77].

In mathematical terms, the synchronization error between the two systems, designated as system (1) and system (25), is defined as follows:

$$e = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} V_1 - U_1 \\ V_2 - U_2 \end{pmatrix} \tag{26}$$

We aim to demonstrate that the discrepancy tends to zero as time approaches z^*. This is accomplished by substituting the expression derived from equation (1) into the error system delineated in equation (25), i.e.,

$$\begin{cases} \dfrac{\partial e_1}{\partial z} = n_1 \Delta e_1 - \left[U_1^2 + V_1^2 + U_1 V_1 \right. \\ \qquad \left. - (a+1)\left(U_1 + V_1 - \frac{a}{a+1}\right)\right] e_1 - e_2 + C, r \in \Omega, \\ \dfrac{\partial e_2}{\partial z} = n_2 \Delta e_2 + c\left(be_1 - e_2\right), r \in \Omega,\ z > 0, \\ \dfrac{\partial e_1}{\partial \eta} = \dfrac{\partial e_2}{\partial \eta} = 0, r \in \partial\Omega, z > 0, \\ e_1(r,0) = V_{1,0}(r) - U_{1,0}(r), \\ e_2(r,0) = V_{2,0}(r) - U_{2,0}(r), r \in \Omega. \end{cases} \tag{27}$$

The control law C is engineered to achieve finite-time synchronization in reaction-diffusion systems, ensuring that spatially distributed components quickly converge to a unified state. This is crucial for applications like autonomous vehicles and sensor networks, where precise and timely coordination is imperative. The control strategy focuses on adjusting the error dynamics between the primary and response systems, driving the error to zero within a finite time using a Lyapunov-based approach. Streamlined for cost-efficiency and simplicity, the control law is particularly suitable for practical implementation in large-scale systems [78], [79].

However, its effectiveness hinges on assumptions such as a homogeneous control environment, accurate system modeling, and tolerable external disturbances. Constraints on control input and the presence of noise could impact performance. Despite these challenges, the control law is expected to significantly enhance system reliability, precision, and overall performance in real-time applications [80], [81].

Theorem 2. *[82] (e_1^*, e_2^*) is a finite-time stability equilibrium point of the nonlinear system (27) if there exists a positive definite Lyapunov function $V : [0, +\infty) \times \Omega \to \mathbb{R}_+$, three class $\mathcal{M}$ functions η, β, Λ, and $\delta > 0$ such that*

1) $\eta \|e(z)\| \leq V(z, e(z)) \leq \beta \|e(z)\|$,
2) $\dfrac{\partial V(z, e(z))}{\partial z} < -\Lambda V(z, e(z))$,
3) $\displaystyle\int_0^{\varepsilon^*} \frac{dz}{\Lambda(z)} < +\infty, (\forall \varepsilon : 0 < \varepsilon^* \leq \delta^*)$.

Definition 2. *[83] If there exists a setting time $z^* > 0$ such that*

$$\lim_{z \to z^*} \|e_1(z)\| + \|e_2(z)\| = 0 \tag{28}$$

and

$$\|e_1(z)\| + \|e_2(z)\| \equiv 0, \quad \forall z \geq z^*, \tag{29}$$

then the derive-response systems (1) and (25) are synchronized in finite time.

Theorem 3. *The derived-response systems, denoted by* (1) *and* (25)*, will achieve stable and synchronized states within a finite time if the following conditions are satisfied:*

$$\Lambda = 2\min\left\{n_1\Upsilon_1 - \frac{1+cb}{2}, n_2\Upsilon_2 + c - \frac{1+cb}{2}\right\} > 0, \tag{30}$$

and by implementing the one-dimensional linear control law

$$C = -\left(3Q^2 + 2\left(a+1\right)Q + a\right)e_1. \tag{31}$$

The finite-time synchronization is defined as

$$z_2^* = \frac{V\left(0\right)}{\Lambda V\left(z_{\max}\right)}. \tag{32}$$

Proof. *We employ Lyapunov's direct method in conjunction with a positive definite Lyapunov function. Let*

$$V\left(z\right) = \frac{1}{2}\int_\Omega e_1^2 + e_2^2\, dr. \tag{33}$$

Utilizing Lemma 1 and 2 yields

$$\begin{aligned}
&\frac{\partial V\left(z\right)}{\partial z} = \int_\Omega e_1\frac{\partial e_1}{\partial z} + e_2\frac{\partial e_2}{\partial z}dr \\
&= \int_\Omega e_1\Bigg[n_1\Delta e_1 - \Bigg(U_1^2 + V_1^2 + U_1V_1 - \left(a+1\right)\big(U_1 \\
&\quad + V_1 - \frac{a}{a+1}\big)\Bigg)e_1 - e_2 - \left(3Q^2 + 2\left(a+1\right)Q + a\right) \\
&\quad e_1\Bigg]dr + \int_\Omega e_2\left[n_2\Delta e_2 + c\left(be_1 - e_2\right)\right]dr \\
&\leq -n_1\int_\Omega\left|\nabla e_1\right|^2 dr - n_2\int_\Omega\left|\nabla e_2\right|^2 dr \\
&\quad + \int_\Omega\Bigg[\left|U_1\right|^2 + \left|V_1\right|^2 + \left|U_1\right|\left|V_1\right| + \left(a+1\right)\big(\left|U_1\right| \\
&\quad + \left|V_1\right| + \frac{a}{a+1}\big)\Bigg]e_1^2 dr + \left(cb-1\right)\int_\Omega e_1e_2 dr \\
&\quad - \left(3Q^2 + 2\left(a+1\right)Q + a\right)\int_\Omega e_1^2 dr - c\int_\Omega e_2^2 dr \\
&\leq -\left(n_1\Upsilon_1 - \frac{1+cb}{2}\right)\int_\Omega e_1^2 dr \\
&\quad - \left(n_2\Upsilon_2 + c - \frac{1+cb}{2}\right)\int_\Omega e_2^2 dr \\
&\leq -2\min\left\{n_1\Upsilon_1 - \frac{1+cb}{2}, n_2\Upsilon_2 + c - \frac{1+cb}{2}\right\}V\left(z\right).
\end{aligned}$$

Let

$$\Lambda = 2\min\left\{\mathfrak{n}_1\Upsilon_1 - \frac{1+\mathfrak{cb}}{2}, \mathfrak{n}_2\Upsilon_2 + \mathfrak{c} - \frac{1+\mathfrak{cb}}{2}\right\}.$$

We have

$$\int_0^{\varepsilon^*}\frac{1}{\Lambda}dz = \frac{\varepsilon^*}{2\min\left\{\mathfrak{n}_1\Upsilon_1 - \frac{1+\mathfrak{cb}}{2}, \mathfrak{n}_2\Upsilon_2 - \mathfrak{c} + \frac{1+\mathfrak{cb}}{2}\right\}} < +\infty. \tag{34}$$

By applying Theorem 2, we can confirm that the zero solution of the error system (27) *signifies the finite-time stability of the equilibrium point* $(e_1^*, e_2^*) = (0,0)$. *Hence, the function* $V\left(z\right)$ *is decreasing and positive when* $0 \leq z < z_2^* \leq z_{\max}$, *and we obtain* $V\left(z\right) > V_2\left(z_{\max}\right)$. *Additionally, we have*

$$\begin{aligned}
V_2\left(z\right) &\leq V_2\left(0\right) - \Lambda\int_0^z V\left(s\right)\, ds \\
&\leq V_2\left(0\right) - \Lambda\int_0^z V\left(v_{\max}\right)\, ds \\
&= V_2\left(0\right) - \Lambda V\left(z_{\max}\right)z.
\end{aligned} \tag{35}$$

As z *approaches a critical value* z_2^*, *where the function* $V\left(z\right)$ *converges to zero, we have*

$$\lim_{z\to z_2^*} V\left(z\right) \leq V\left(0\right) - \Lambda V\left(z_{\max}\right)z_2^* = 0. \tag{36}$$

So, we conclude that the finite-time synchronization is defined as

$$z_2^* = \frac{V\left(0\right)}{\Lambda V\left(z_{\max}\right)}.$$

Consequently, the derived-response systems (1) *and* (25) *achieve synchronization within a finite time.*

V. Numerical Simulations

In this section, we present practical simulations designed to elucidate the theoretical principles discussed earlier. The simulations were conducted using the finite difference method implemented in MATLAB. FDM is a numerical approach used to solve differential equations by discretizing continuous variables, such as time and space, into a grid. The grid size and time step are crucial elements that determine the resolution and stability of the solution. In spatial discretization, the domain is divided into a grid of points, with smaller grid sizes providing higher resolution but increasing computational cost. Time is similarly discretized into intervals, with the time step affecting both the stability and accuracy of the method. Ensuring stability often requires the time step to meet specific criteria, such as the Courant-Friedrichs-Lewy (CFL) condition. When applied to finite-time stability, FDM approximates the derivatives in the governing differential equations, and the chosen grid and time step must preserve the stability properties of the continuous system. For finite-time synchronization, the method evaluates synchronization conditions for coupled systems, where accuracy depends on minimizing the discretization error. Overall, careful selection of grid size and time step is essential to ensure that the numerical solution accurately represents the dynamics of the continuous system within a finite time frame. This section includes two illustrative examples for completeness.

Example 1. We examine the domain $\Omega = [0,15]$, with $z \in [0,1]$, and define the parameters as given in Table I. The initial

TABLE I. PARAMETER VALUES

Variable	Value
n_1	1
n_2	1
a	0.25
b	0.76
c	3.5
Q	1.5
δ	3
ε	17.6192
Υ_1	9.32693
Υ_2	9.32693
$\mathbb{N}$	100

conditions are given by

$$\begin{cases} U_{1,0}(r) = 1.5, \\ U_{1,0}(r) = 1.5. \end{cases} \tag{37}$$

The condition of Theorem 1 is satisfied

$$M = \max\left\{3Q^2 + 2(a+1)Q + a + \frac{cb+1}{2} - n_1\Upsilon_1, \frac{cb+1}{2} - c - n_2\Upsilon_2\right\} = 3.2531, \tag{38}$$

and the finite-time stability is estimated as

$$z_1^* = \frac{2}{M}\ln\left(\frac{\varepsilon}{\sqrt{2}\delta}\right) = 0.8754s. \tag{39}$$

According to Theorem 1, these solutions exhibit finite-time stability at the equilibrium point $(U_1^*, U_2^*) = (0, 0)$. The figures and their interpretations are summarized in Table II.

TABLE II. SUMMARY OF SYSTEMS ANALYSIS AND FIGURES

Description	Figures
The temporal and spatial solutions of system (1) with homogeneous Neumann boundary conditions are shown.	Fig. 1
To numerically validate the finite-time stability, temporal and spatial solutions of the system, along with error estimates, are presented for a one-dimensional space.	Fig. 2
This demonstration confirms that the errors converge to 0 as z approaches $z_1^* = 0.8754$ seconds, validating the theoretical predictions.	Fig. 3

A. Sensitivity Analysis of Finite-Time Stability

To explore the sensitivity of the system to changes in parameter values, we analyze how variations in key parameters affect the finite-time stability. We consider perturbations in the parameters a, b, and c, and observe their impact on the finite-time stability time z_1^*. For each parameter, we vary its value within a range and compute the corresponding finite-time stability time. The results are summarized in Table III. The sensitivity analysis reveals that changes in c have minimal to moderate impacts on the finite-time stability time, while variations in a and b can lead to significant changes. This

TABLE III. SUMMARY OF SYSTEM ANALYSIS AND FIGURES

Parameter	Perturbation Range	Impact on z_1^*	Remarks
a	[0.16175, 0.25]	0.8754s - 1s	Significant impact
b	[0.6398, 0.76]	0.8754s - 1s	Moderate impact
c	[2.1035, 3.5]	0.8754s - 1s	Minimal impact

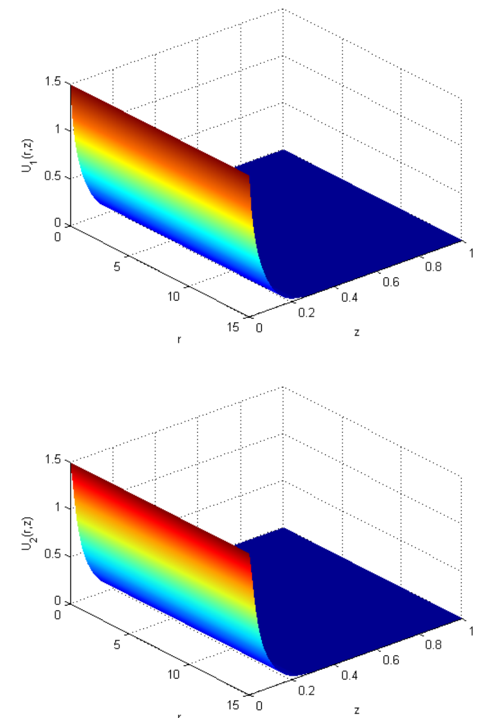

Fig. 1. Dynamic behavior of solution $U_1(r, z)$ and $U_2(r, z)$.

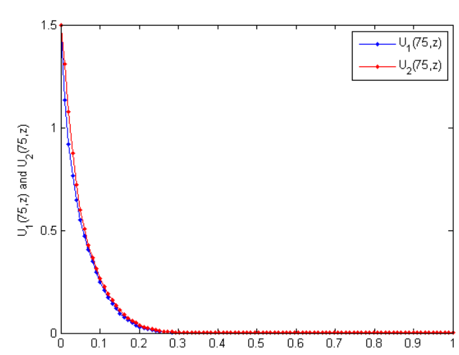

Fig. 2. The state trajectories of the solutions $U_1(75, z)$ and $U_2(75, z)$.

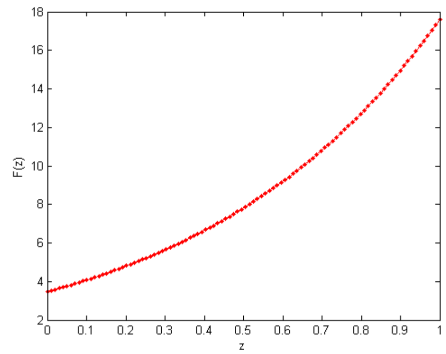

Fig. 3. Estimation of the function $F(z)$.

information is crucial for designing robust control strategies that can accommodate parameter uncertainties.

Example 2. Consider the domain Ω, defined as the interval $[0,6]$, with z ranging within $[0,100]$ in Table IV. The initial

TABLE IV. PARAMETER VALUES

Variable	Value
n_1	13
n_2	13
a	1.4
b	1.5
c	1.3294
Q	0.5404
Υ_1	10^6
Υ_2	10^6
$\mathbb{N}$	100

conditions for the derived-response systems (1) and (25) are set as follows:

$$\begin{cases} U_{1,0}(r) = 0.5, \\ U_{2,0}(r) = 0.5, \end{cases} \tag{40}$$

and

$$\begin{cases} V_{1,0}(r) = 1, \\ V_{2,0}(r) = 1. \end{cases} \tag{41}$$

Based on the conditions of Theorem 3, we obtain

$$\Lambda = \min\left\{ n_1\Upsilon_1 - \frac{1+cb}{2}, n_2\Upsilon_2 + c - \frac{1+cb}{2} \right\} \tag{42}$$

$$= 2.6 \times 10^7. \tag{43}$$

As a consequence, the one-dimensional linear control law is defined as

$$C = -4.87e_1 \tag{44}$$

and

$$V(0) = 25, \quad V(100) = 9.6281 \times 10^{-9}. \tag{45}$$

Thus, we conclude

$$z_2^* = \frac{V(0)}{\Lambda V(z_{\max})} = 99.8679s. \tag{46}$$

Consequently, the Lyapunov function converges to zero as z approaches $z_2^* = 99.8679$ s, according to Theorem 3, demonstrating that the derived-response systems (1)-(25) exhibit finite-time synchronization at $z_2^* = 99.8679$ s. Table V summarizes the figures and their interpretations.

TABLE V. SUMMARY OF SYSTEM ANALYSIS AND FIGURES

Description	Figures
The spatiotemporal dynamics of derive-response systems (1)-(25) are illustrated, with additional insights in both two-dimensional and one-dimensional spaces.	Figs. 4, 5, 7
The spatiotemporal solutions of the error synchronization system (27) are shown through numerical simulations, suggesting that errors converge to 0 as z approaches $z_2^* = 99.8679s$, demonstrating finite-time behavior.	Figs. 6, 7
The Lyapunov function converges to zero as z approaches $z^* = 99.8679s$.	Fig. 8

B. Error Estimation and Analysis

The error estimation and analysis shown in Table VI

TABLE VI. ERROR ESTIMATION AND ANALYSIS

Error $e_1 \times 10^{-6}$	Error $e_2 \times 10^{-5}$
-9.266181128	4.1680331290
-8.280231685	3.7246841735
-7.399267522	3.3284975276
-6.612093074	2.9744549606
-5.90870968	2.6580724948
-5.280187165	2.3753434684
-4.718549264	2.1226876756
-4.216671364	1.8969059356
-3.768189265	1.6951395083
-3.367417753	1.5148338399
-3.009277928	1.3537061736

C. Sensitivity Analysis of Finite-Time Synchronization:

To assess how variations in key parameters influence the results, we conduct a sensitivity analysis. This analysis examines how alterations in critical parameters affect the outcome, as summarized in Table VII.

TABLE VII. SENSITIVITY ANALYSIS RESULTS OF FINITE-TIME SYNCHRONIZATION

Parameter	Perturbation Range	Impact on z_2^*	Remarks
a	[1.3996145, 1.4]	99.8679s - 100s	Minimal impact
b	[1.5, 1.500375]	99.8679s - 100s	Moderate impact
c	[1.3294, 1.3294714]	99.8679s - 100s	Significant impact

The results of the sensitivity analysis indicate that while fluctuations in parameters a and b lead to only minor or moderate changes in the finite-time stability time z_2^*, variations in parameter c can cause considerable adjustments. Understanding these effects is crucial for developing control strategies that can effectively handle parameter variability and ensure system robustness.

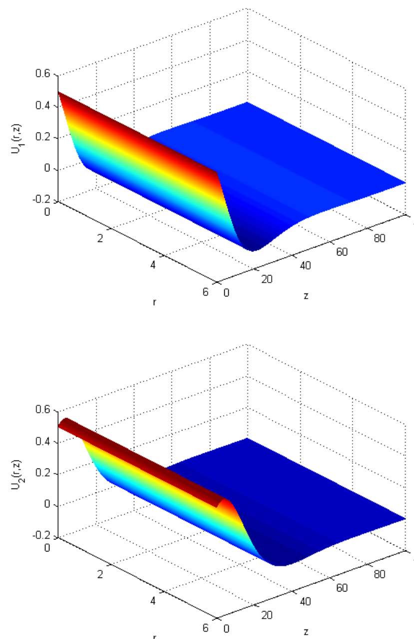

Fig. 4. Dynamic behavior of the master system (1): $U_1(r, z)$ and $U_2(r, z)$.

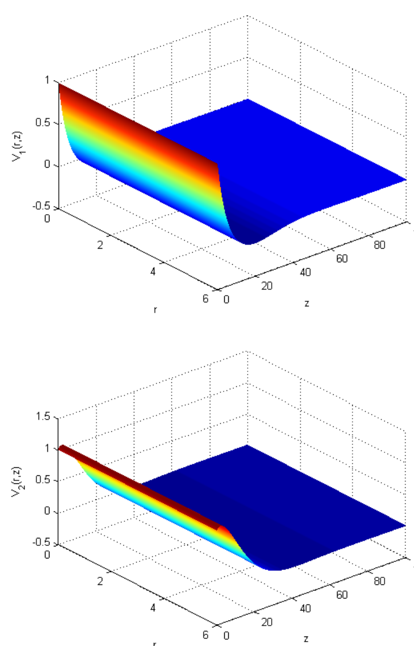

Fig. 5. Dynamic behavior of the slave system (25): $V_1(r, z)$ and $V_2(r, z)$.

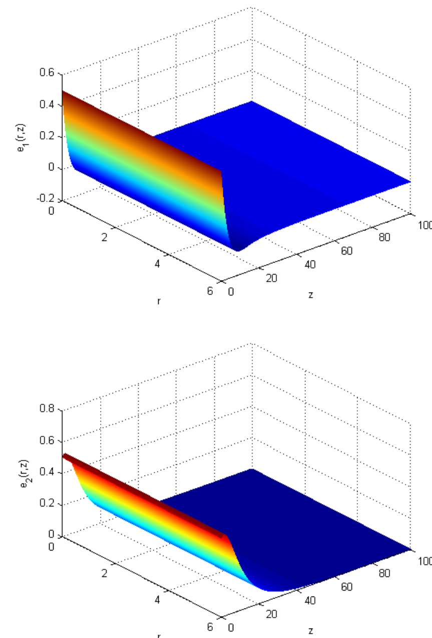

Fig. 6. Dynamic behavior of the error system (27): $e_1(r, z)$ and $e_2(r, z)$.

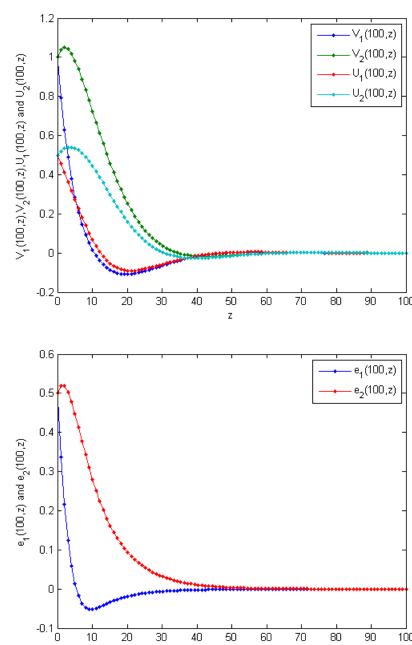

Fig. 7. Solutions of the master-slave systems (1), (25) and the error system (27) at $r = 100$.

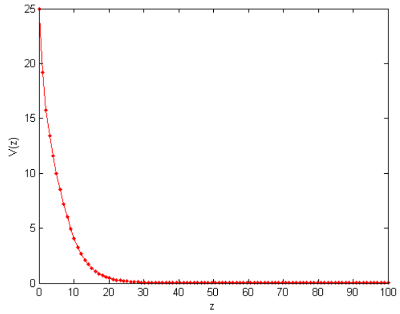

Fig. 8. Estimation of the Lyapunov function $V(z)$.

VI. Conclusion

In this manuscript, we have investigated the finite-time stability of the FitzHugh-Nagumo reaction-diffusion system, providing a comprehensive analysis that enhances our understanding of rapid stabilization in such systems. We have established a criterion for finite-time stability, which is essential for applications requiring quick and reliable stabilization. This criterion, along with explicit formulas for stability parameters, is particularly relevant for real-time control systems, sensor networks, and industrial automation, where prompt and predictable behavior is crucial.

Our study also extends to finite-time synchronization schemes, emphasizing their importance in achieving coherent dynamics across spatially distributed components. This is especially relevant for precision-driven fields such as robotics and aerospace, where synchronization within a finite time frame can improve accuracy and operational efficiency. The numerical simulations performed using the finite difference method (FDM) in MATLAB have provided valuable insights into the FitzHugh-Nagumo reaction-diffusion system. These simulations were instrumental in validating our theoretical claims. For instance, in Example 1, the system's demonstrated finite-time stability at the equilibrium point corroborates our theoretical criterion, as errors converged to zero in line with the predicted stability parameter z_1^*. This validation supports the robustness of our theoretical framework, showcasing its applicability to practical scenarios. Sensitivity analysis from the simulations also aligned with our theoretical predictions, indicating that variations in parameters a and b significantly impact stability, while parameter c has minimal effects. Similarly, in Example 2, our analysis of the error synchronization system affirmed the theoretical expectations of finite-time behavior, with system errors converging to zero at z_2^*. This outcome not only validates our theoretical results but also highlights the practical relevance of our approach. The sensitivity analysis from these simulations showed consistent results with our theoretical predictions, emphasizing the significant impact of parameter c on the stability time and the relatively minor effects of parameters a and b.

A critical aspect of this study is the comparison of our results with existing literature. While previous studies have explored various aspects of synchronization and stability in reaction-diffusion systems, our work introduces a novel finite-time stability criterion specific to the FitzHugh-Nagumo model. Unlike the asymptotic stability approaches commonly found in the literature, our focus on finite-time behavior offers more immediate and practical insights for applications requiring rapid system stabilization. The explicit formulas for stability parameters presented here provide a more direct and applicable framework compared to earlier theoretical works that often require more complex or abstract methods.

Future research could extend the finite-time stability and synchronization analysis to other reaction-diffusion systems and higher-dimensional models. Investigating the effects of nonlinearities and external perturbations on finite-time stability could provide deeper insights into system robustness. Additionally, incorporating adaptive control strategies that dynamically adjust parameters in response to varying conditions would enhance the practical applicability of the proposed methods. Exploring these areas will offer further understanding and broaden the applicability of finite-time stability and synchronization techniques. A limitation of this study is that the theoretical results and numerical simulations are based on specific parameter ranges and system configurations. Future work could address this by exploring a wider range of parameters and more complex system configurations. Although the sensitivity analysis provides valuable insights, it is based on idealized perturbations; real-world applications may require more nuanced analyses.

This work contributes to the field of reaction-diffusion systems by providing new theoretical insights into finite-time stability and synchronization. The proposed criterion and stability parameter formulas, combined with comprehensive numerical simulations, offer a robust framework for designing and controlling systems requiring rapid stabilization and synchronization. The study underscores the importance of careful parameter selection and sensitivity analysis in ensuring system stability and effectiveness, paving the way for future advancements in this area. By comparing our findings with previous research and validating our theoretical predictions through simulations, we have demonstrated the novelty and enhanced practicality of our approach, thus providing a meaningful contribution to the ongoing development of synchronization strategies in complex systems.

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