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Computational Approaches to Two-Energy Group Neutron Diffusion in Cylindrical Reactors

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***Abstract*—This study addresses the critical need for accurate neutron diffusion modeling in cylindrical reactors, focusing on the two-energy groups neutron diffusion system. Such modeling is essential for optimizing reactor design and safety in nuclear engineering. The research primarily aims to enhance computational methods by transitioning from a traditional integer-order model to a more sophisticated fractional-order model, which can capture complex physical phenomena with greater precision. The study employs the Laplace Transform Method (LTM) to first solve the integer-order system and then extends this approach to a fractional-order system using the Caputo derivative, a method well-suited for systems with memory effects. To efficiently solve the resulting fractional-order model, we introduce the Modified Fractional Euler Method (MFEM), designed to improve numerical accuracy and stability. The effectiveness of this approach is demonstrated through specific numerical applications, such as simulating neutron flux distributions, which validate the model's accuracy and its potential impact on advancing reactor physics. These applications showcase the practical relevance of the proposed methods and their contribution to improving nuclear reactor simulations.**

Keywords*—*Two–Energy Groups of a Neutron Diffusion System; Cylindrical Reactors; Laplace Transform Method; Fractional Calculus; Modified Fractional Euler Method; Caputo Fractional Derivative.

I. Introduction

The field of nuclear physics has received a great deal of interest among scientists, especially for discovering the behaviors of nuclear reactors. This interest is justified by the various results obtained by modeling these phenomena using mathematical equations. Among the most significant equations is the neutron diffusion model, which can be used to describe how the neutrons operate in nuclear reactors and in many physical applications [1]. For this purpose, a several analytical and numerical methods are developed to solve such problems. To cite a few, a solution of the neutron diffusion equation in hemispherical symmetry using the homotopy perturbation method is given in [2]. The authors in [3] have proposed a numerical techniques for dealing with the neutron diffusion equations in the nuclear Reactors. In [4], the exact solution of the one-dimensional neutron diffusion kinetic equation with one delayed precursor concentration in Cartesian geometry is provided. As for further research, see [5]–[8] .

More specifically, the effects of many energy-level interactions can be taken into account in the time-dependent problem neutron diffusion model for multi-energy groups, providing a more comprehensive knowledge of neutron behavior. Therefore, how to solve these systems has become an important contribution and a major challenge among researchers to develop new methods in this context [9]. For example in [10], by using the residual power series method, a multi-energy groups of a neutron diffusion equations system is analytically solved in three different reactors: slab, cylinder and sphere.

In [11], a new package of codes was proposed to simulate a three-dimensional reactor cores by solving the neutron diffusion equations. Such codes are based on the nodal expansion methodm which can calculate each node's neutron flux/power distribution. In [12], an APR-1400 reactor core analysis was performed by using the well-known two–step method, which was applied to the APR-1400 first cycle using the open-source nodal diffusion code. In [13], a three dimensional diffusion theory code was developed based on the nodal expansion method. Such a code uses the quartic flux expansion with quadratic transverse leakage approximation. Also, it is mainly used for calculation of keff, neutron flux and integral kinetics parameters. In [14], several challenges encountered in the development of nodal expansion method in cylindrical geometry and the method to circumvent these difficulties was introduced and discussed. Very recently in [15], M. Shqair et al. presented a solution for the multi-group reflected spherical reactor system of equations using the Homotopy perturbation method. In a particular case, recent research has concentrated on the two–energy groups neutron diffusion model because of its numerous

applications, for more details see [16]–[19] and references therein.

On the other hand, some mathematicians have studied such problems in the fractional calculus, where this field has several applications in different disciplines especially in the modeling of physical phenomena where we simulate systems having memory effects [20]. Diffusion can behave unconventionally in certain situations, making it impossible for the classical diffusion theory to predict diffusion effectively. Fractional calculus, in which the derivative's order is represented by a fractional-order value, can be used to analyze and simulate these anomalous diffusion processes; see, for instance, the research works [1], [17], [21], [22]. In addition, the reader may take a full overview about the fractional calculus and its applications by referring to the references [23]–[53].

Choosing between a two–energy groups neutron diffusion model and a multi-energy group neutron diffusion model for simulating cylindrical reactor fluxes involves a trade-off between model accuracy and computational efficiency. Here are some reasons why one might choose the simpler two–energy groups model over a more complex multi-energy group model:

- **Reduced computational complexity:** Multi-energy group models involve solving neutron transport equations for multiple energy ranges, leading to a larger system of equations and increased computational complexity. Two–energy groups models are computationally less demanding, making them more suitable for certain applications, especially when quick simulations or iterative calculations are required.
- **Parameter estimation and calibration:** The two–energy groups model is easier to parameterize and calibrate. With fewer energy groups, there are fewer cross-section data points to be determined or measured experimentally. This simplifies the calibration process and reduces uncertainties associated with cross-section data.
- **Data availability:** In some cases, comprehensive cross-section data for all energy groups may not be readily available or may be associated with higher uncertainties. A two–energy groups model can be more practical when data limitations or uncertainties are a concern.
- **Sensitivity studies:** For sensitivity studies or preliminary analyses, a two–energy groups model can provide a quick overview of the reactor behavior. This allows researchers or analysts to identify key factors influencing the system before committing to the computational demands of a more detailed multi-energy group model.

In light of the aforementioned discussion, the main objective of this work is to solve an integer-order two–energy groups neutron diffusion model in a cylindrical reactor using the Laplace Transform Method (LTM), where the cylindrical radius r would be taken into account as a time domain in order to achieve this. The LTM method performs perfectly and is appropriate for solving this kind of problem without discretization or perturbation. The next part of this research is concerned with the solution of the fractional order system related to the integer order one, where this result is achieved by developing a technique to reduce the 2β-fractional-order system into another fractional-order system of order β, where $0 < \beta \leq 1$. To put it briefly, through this reduction, the problem is simplified and becomes amenable to numerical methods and computational methods. Then, we apply one of the recent modifications of the fractional Euler method called the Modified Fractional Euler Method (MFEM) to solve the final system see [54]. At last, a numerical simulations are presented to demonstrate the performance of the scheme.

II. Basic Concepts

First, a brief overview on the fractional calculus applied in this article will be provided in the following section. The reader may also refer to the references [55]–[58] to get a full description about the fractional-order operators in all.

Definition 1: [59], [60] The Riemann-Liouville fractional integral of order $\beta \geq 0$ for a continuous function ψ on $[0, b]$ is defined by

$$J^{\beta}\psi(t) = \begin{cases} \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1}\psi(\tau)d\tau, & \beta > 0 \\ \psi(t), & \beta = 0 \ , \end{cases}$$

where $0 < t \leq b$, and $\Gamma(\cdot)$ is the Gamma function.

Definition 2: [59], [60] For a function $\psi \in C^m([0, b], \mathbb{R})$, $m \in \mathbb{N}^*$ and $m - 1 < \beta \leq m$, the operator D^{β} is the Caputo fractional derivative given by

$$\begin{aligned} D^{\beta}\psi(t) &= J^{m-\beta}D^m\psi(t) \\ &= \frac{1}{\Gamma(n-\beta)} \int_0^t (t-\tau)^{m-\beta-1}\psi^{(m)}(\tau)d\tau, \end{aligned}$$

where $t > 0$, and m is the smallest integer number greater than β.

At this point, we list certain properties associated to the previous two operators and some important connections between them, see [59], [60].

Proposition 1: The fractional integral of Riemann-Liouville has the following properties:

1) $J^{\beta}t^{\alpha} = \frac{\Gamma(\alpha+1)}{\Gamma(\beta+\alpha+1)}t^{\beta+\alpha}, \quad \alpha \geq -1,$
2) $J^{\beta}J^{\alpha}\psi(t) = J^{\alpha+\beta}\psi(t) = J^{\alpha}J^{\beta}\psi(t), \quad \beta, \alpha \geq 0.$

Proposition 2: The fractional derivative of Caputo fulfills the following properties:

1) $D^{\beta}\mathcal{C} = 0,$ where $\mathcal{C} \in \mathbb{R}.$
2) $D^{\beta}t^{\alpha} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\beta+1)}t^{\alpha-\beta},$ where $\alpha > \beta - 1.$
3) $D^{\beta}(\lambda_1\psi_1(t) + \lambda_2\psi_2(t)) = \lambda_1 D^{\beta}(\psi_1(t)) + \lambda_2 D^{\beta}(\psi_2(t)),$

where λ_1 and λ_2 are constants.

Proposition 3: If $m-1 < \beta \leq m$, $m \in \mathbb{N}$, the main connections between the Riemann-Liouville integral and the Caputo derivative are defined as follows:

1) $D^{\beta} J^{\beta} \psi(t) = \psi(t)$.
2) $J^{\beta} D^{\beta} \psi(t) = \psi(t) - \sum_{k=1}^{m-1} \psi^{k}(0^{+}) \frac{t^{k}}{k!}$.

In addition, the following basic theorem called generalized Taylor's formula is required for our main findings. It has been utilized a lot of times recently in many research papers due to its great importance in many applications, see [61]–[63].

Theorem 1: [64] Suppose that $D^{k\beta}\psi(t)$ is a continuous function on $(0, b]$, for $k = 0, 1, 2, \cdots, m+1$, where $0 < \beta \leq 1$. Then, The function ψ for the node t_0 can then be expanded as follows:

$$\begin{aligned} \psi(t) = & \sum_{i=0}^{m} \frac{(t-t_0)^{i\beta}}{\Gamma(i\beta+1)} D^{i\beta}\psi(t_0) \\ & + \frac{(t-t_0)^{(m+1)\beta}}{\Gamma((m+1)\beta+1)} D^{(m+1)\beta}\psi(\eta), \end{aligned} \tag{1}$$

with $0 < \eta \leq t$, $\forall t \in (0, b]$.

In the second part of this section, we will introduce the Laplace transform with some of its properties since it is an important factor in this article.

Definition 3: [65] The Laplace transform of a function ψ defined for $0 \leq t < \infty$ is another function denoted $\mathcal{T}(s)$, which can be presented as

$$\mathcal{T}(s) = \mathscr{L}\{\psi\} := \int_{0}^{\infty} \psi(t) e^{-st} dt. \tag{2}$$

Proposition 4: [65] Among the properties of the Laplace transform we mention:

1) $\mathscr{L}\{t\psi\} = -\frac{d}{ds}\mathscr{L}\{\psi\}$.
2) If the function ψ is a continuous function on $[0, +\infty[$, piecewise differentiable on $]0, +\infty[$ where whose derivative is piecewise continuous, then

$$\mathscr{L}\{\psi'(t)\} = -\psi(0^{+}) + s\mathscr{L}\{\psi\} = s\mathcal{T}(s) - \psi(0^{+}).$$

3) Also, if the ψ' is a continuous function on $\mathbb{R}_{+}^{*}$, piecewise differentiable on $\mathbb{R}_{+}^{*}$ where whose derivative is piecewise continuous, then

$$\mathscr{L}\{\psi''(t)\} = s^{2}\mathcal{T}(s) - s\psi(0^{+}) - \psi'(0^{+}),$$

where $\psi(0^{+}) = \lim_{t \longrightarrow 0^{+}} \psi(t)$.

Next, let us recall the aforesaid MFEM, which is defined in the reference [54], and it was applied in order to solve the following problem:

$$\begin{cases} D^{\beta}\psi(t) = g(t, \psi(t)), \quad t \in [a, b], \\ \psi(a) = \psi_0, \end{cases} \tag{3}$$

where $0 < \beta \leq 1$, D^{β} denotes the Caputo derivative and $g : [a, b] \times \mathbb{R} \to \mathbb{R}$.

In this context, we divide $[a, b]$ into $a = t_0 < t_1 = t_0 + h < \cdots < t_m = t_0 + mh = b$, where $h = \frac{b-a}{m}$ is the step size corresponds to the mesh points $t_i = t_0 + ih$, $i = 1, \ldots, m$. Thus, by applying the first three terms of Theorem 1, we can acquire

$$\begin{aligned} & v_0 = \psi_0, \\ & v_{i+1} = v_i + \frac{h^{\beta}}{\Gamma(\beta+1)} \\ & \quad g\left(t_i + \frac{h^{\beta}}{2\Gamma(\beta+1)}, v_i + \frac{h^{\beta}}{2\Gamma(\beta+1)} g(t_i, v_i)\right), \end{aligned} \tag{4}$$

where v_i denotes the numerical solution of problem (3), for $i = 1, 2, \cdots, m-1$. The reader may refer to the references [66], [67] to see how one can use this scheme for dealing with different problems.

III. Two–Energy Groups of Neutrons Cylindrical Reactor

In this section, we propose to solve the integer-order two–energy groups of neutron diffusion system for cylindrical reactors by applying LTM. Then, its associated fractional order model will be solved numerically by the use of the MFEM.

A. Solving the Integer-Order Model

At first, it is supposed that the neutron diffusion system with two–energy groups has a unique solution in the integration interval, where the classical form of this model characterizes the behavior of the neutrons in nuclear reactors, where each flux exhibits the neutron flux with a specific speed, so that it is given as follows [68], [69]:

$$\begin{cases} \nabla^{2}\psi_1(r) + \mathcal{C}_{11}\psi_1(r) + \mathcal{C}_{12}\psi_2(r) = 0, \\ \nabla^{2}\psi_2(r) + \mathcal{C}_{21}\psi_1(r) + \mathcal{C}_{22}\psi_2(r) = 0, \\ \psi_i(0^{+}) = \mathcal{A}_i, \quad \psi_i'(0^{+}) = \mathcal{B}_i, \ i = 1, 2, \end{cases} \tag{5}$$

where the fluxes $\psi_i(r)$ are functions of independent variable r $(r > 0)$, and $\mathcal{A}_i, \mathcal{B}_i \in \mathbb{R}$, then the connection between fluxes in different energy groups of neutrons is denoted by constant $\mathcal{C}_{ij}$, and the group buckling is referred by the constant C_{ii}, which can be defined by

$$\begin{cases} \mathcal{C}_{ij} = \frac{\sum\sum_{sij} + x_i v_j \sum_{fi}}{\mathcal{D}_i}, \\ \mathcal{C}_{ii} = \frac{x_i v_i \sum_{fi} - (\sum_{\gamma i} + \sum\sum_{sij})}{\mathcal{D}_i}, \end{cases} \tag{6}$$

such that $\mathcal{D}_i$ is a group diffusion coefficient that can be defined as follows:

$$D_i = \frac{1}{3(\sum_{fi} + \sum_{sii} + \sum\sum_{sij} + \sum_{\gamma i})}, \tag{7}$$

for $i, j = 1, 2$. All the constants mentioned in (6) and (7) are identified in the field of neutrons fissions that are emitted with energies.

In the cylindrical reactor geometry, the Laplacian operator for $\psi_1(r)$ and $\psi_2(r)$ becomes:

$$\begin{cases} \nabla^2\psi_1(r) &= \frac{\partial^2}{\partial r^2}\psi_1(r) + \frac{1}{r}\frac{\partial}{\partial r}\psi_1(r). \\ \\ \nabla^2\psi_2(r) &= \frac{\partial^2}{\partial r^2}\psi_2(r) + \frac{1}{r}\frac{\partial}{\partial r}\psi_2(r). \end{cases} \tag{8}$$

Based on (8), the system (5) can be presented in the following format:

$$\begin{cases} r\psi_1''(r) + \psi_1'(r) + r\mathcal{C}_{11}\psi_1(r) + r\mathcal{C}_{12}\psi_2(r) = 0, \\ \\ r\psi_2''(r) + \psi_2'(r) + r\mathcal{C}_{21}\psi_1(r) + r\mathcal{C}_{22}\psi_2(r) = 0, \\ \\ \psi_i(0^+) = \mathcal{A}_i, \quad \psi_i'(0^+) = \mathcal{B}_i, \ i = 1, 2. \end{cases} \tag{9}$$

In the passage that follows, the integer-order two–energy groups of neutron diffusion model (9) will be solved in cylindrical reactors via LTM and by allowing the cylindrical radius r to be a time domain. First, we apply the Laplace transform to each term of the equations associated to (9), we obtain:

$$\begin{aligned} &\mathscr{L}(r\psi_1''(r)) + \mathscr{L}(\psi_1'(r)) \\ &\qquad + \mathcal{C}_{11}\mathscr{L}(r\psi_1(r)) + \mathcal{C}_{12}\mathscr{L}(r\psi_2(r)) = 0, \\ &\mathscr{L}(r\psi_2''(r)) + \mathscr{L}(\psi_2'(r)) \\ &\qquad + \mathcal{C}_{21}\mathscr{L}(r\psi_1(r)) + \mathcal{C}_{22}\mathscr{L}(r\psi_2(r)) = 0. \end{aligned} \tag{10}$$

If the functions ψ_i satisfies the conditions mentioned in the Property 4, the above quantities becomes

$$\begin{aligned} &-\frac{d}{ds}\left(\mathscr{L}\{\psi_1''(r)\}\right) + \left(s\mathscr{L}\{\psi_1(r)\} - \psi_1(0^+)\right) \\ &+ \mathcal{C}_{11}\left(\frac{-d}{ds}\mathscr{L}\{\psi_1(r)\}\right) + \mathcal{C}_{12}\left(\frac{-d}{ds}\mathscr{L}\{\psi_2(r)\}\right) = 0, \\ &-\frac{d}{ds}\left(\mathscr{L}\{\psi_2''(r)\}\right) + \left(s\mathscr{L}\{\psi_2(r)\} - \psi_2(0^+)\right) \\ &+ \mathcal{C}_{21}\left(\frac{-d}{ds}\mathscr{L}\{\psi_1(r)\}\right) + \mathcal{C}_{22}\left(\frac{-d}{ds}\mathscr{L}\{\psi_2(r)\}\right) = 0. \end{aligned} \tag{11}$$

As a direct consequence,

$$\begin{aligned} &\frac{-d}{ds}\left(s^2\mathscr{L}\{\psi_1(r)\} - s\psi_1(0^+) - \psi_1'(0^+)\right) \\ &+ \left(s\mathscr{L}\{\psi_1(r)\} - \psi_1(0^+)\right) + \mathcal{C}_{11}\left(\frac{-d}{ds}\mathscr{L}\{\psi_1(r)\}\right) \\ &+ \mathcal{C}_{12}\left(\frac{-d}{ds}\mathscr{L}\{\psi_2(r)\}\right) = 0, \\ &\frac{-d}{ds}\left(s^2\mathscr{L}\{\psi_2(r)\} - s\psi_2(0^+) - \psi_2'(0^+)\right) \\ &+ \left(s\mathscr{L}\{\psi_2(r)\} - \psi_2(0^+)\right) + \mathcal{C}_{21}\left(\frac{-d}{ds}\mathscr{L}\{\psi_1(r)\}\right) \\ &+ \mathcal{C}_{22}\left(\frac{-d}{ds}\mathscr{L}\{\psi_2(r)\}\right) = 0. \end{aligned} \tag{12}$$

Then, under the supposition $\mathscr{L}\{\psi_i(r)\}) = \mathcal{T}_i(s)$, for $i = 1, 2$, we get:

$$\begin{cases} -\frac{d}{ds}(s^2\mathcal{T}_1(s) - \mathcal{A}_1 s - \mathcal{B}_1) + (s\mathcal{T}_1(s) - \mathcal{A}_1) \\ -\mathcal{C}_{11}\mathcal{T}_1'(s) - \mathcal{C}_{12}\mathcal{T}_2'(s) = 0, \\ \\ -\frac{d}{ds}(s^2\mathcal{T}_2(s) - \mathcal{A}_2 s - \mathcal{B}_2) + (s\mathcal{T}_2(s) - \mathcal{A}_2) \\ -\mathcal{C}_{21}\mathcal{T}_1'(s) - \mathcal{C}_{22}\mathcal{T}_2'(s) = 0. \end{cases} \tag{13}$$

By some computations and simplifications, the system (13) produces:

$$\begin{cases} -(s^2 + \mathcal{C}_{11})\mathcal{T}_1'(s) - \mathcal{C}_{12}\mathcal{T}_2'(s) = s\mathcal{T}_1(s), \\ \\ -\mathcal{C}_{21}\mathcal{T}_1'(s) - (s^2 + \mathcal{C}_{22})\mathcal{T}_2'(s) = s\mathcal{T}_2(s). \end{cases} \tag{14}$$

The system (14) can be translated to the following matrix form

$$\begin{bmatrix} -(s^2 + \mathcal{C}_{11}) & -\mathcal{C}_{12} \\ -\mathcal{C}_{21} & -(s^2 + \mathcal{C}_{22}) \end{bmatrix} \begin{bmatrix} \mathcal{T}_1'(s) \\ \mathcal{T}_2'(s) \end{bmatrix} = s \begin{bmatrix} \mathcal{T}_1(s) \\ \mathcal{T}_2(s) \end{bmatrix}, \tag{15}$$

which is equivalent to

$$M(s)\mathcal{T}'(s) = s\mathcal{T}(s), \tag{16}$$

where

$$M(s) = \begin{bmatrix} -(s^2 + \mathcal{C}_{11}) & -\mathcal{C}_{12} \\ -\mathcal{C}_{21} & -(s^2 + \mathcal{C}_{22}) \end{bmatrix},$$

and

$$\mathcal{T}'(s) = \begin{bmatrix} \mathcal{T}_1'(s) \\ \mathcal{T}_2'(s) \end{bmatrix}, \mathcal{T}(s) = \begin{bmatrix} \mathcal{T}_1(s) \\ \mathcal{T}_2(s) \end{bmatrix}.$$

Thus, if the matrix $M(s)$ is invertible, we can formulate (16) as follows:

$$\mathcal{T}'(s) = sM^{-1}(s)\mathcal{T}(s). \tag{17}$$

The system (17) can be solved numerically by using a prepared MATLAB code, where this system is considered as a homogeneous linear system with coefficient variables. Therefore, its solution is given in the following form:

$$\mathcal{T}(s) = \mathcal{K}\psi(s), \tag{18}$$

where $\mathcal{K}$ is a two-dimensional vector consisting of arbitrary numbers and the function $\psi(s)$ defined by

$$\psi(s) = e^{\int_0^s \tau M(\tau)d\tau}. \tag{19}$$

Note that by the assumption $\mathscr{L}\{\psi_i(r)\}) = \mathcal{T}_i(s)$, then $\psi_i(r) = \mathscr{L}^{-1}\{\mathcal{T}_i(s)\}$, for all $i = 1, 2$. This would give the solution $\psi_i(r)$ of system (9).

B. Solving the 2β-Fractional Order Model

In this part, the MFEM will be applied to solve the fractional order two–energy groups neutron diffusion system in cylindrical reactors, where this model is constructed after the fractionalization of the system (9) by applying the Caputo fractional

derivative to its equations. In that case, the two-dimensional fractional system would then acquire the following form:

$$\begin{cases} D^{2\beta}\psi_1(r) + \frac{1}{r}D^{\beta}\psi_1(r) + \mathcal{C}_{11}\psi_1(r) \\ \qquad +\mathcal{C}_{12}\psi_2(r) = 0, \\ D^{2\beta}\psi_2(r) + \frac{1}{r}D^{\beta}\psi_2(r) + \mathcal{C}_{21}\psi_1(r) \\ \qquad +\mathcal{C}_{22}\psi_2(r) = 0, \quad r \in I, \\ \psi_i(a) = \mathcal{A}_i, \quad D^{\beta}\psi_i(a) = \mathcal{B}_i, \ i = 1, 2, \end{cases} \tag{20}$$

where $I = [a, b]$, $0 < \beta \leq 1$, $\mathcal{A}_i, \mathcal{B}_i \in \mathbb{R}$ for $i = 1, 2$ and the operator D^{β} is the Caputo derivative. The constants $a, b > 0$, where a is a smallest number close to 0 with positive values. Otherwise, $\psi_i(a) = \psi_i(0^+) = \lim_{t \longrightarrow 0^+} \psi_i(t)$ for $i = 1, 2$.

Now, we note that (20) is a system of 2β fractional order, and for solve this system by MFEM, we shall first reduce it into a one dimensional fractional differential system of order β which they are equivalent. To do this conversation, we introduce the next lemma [9].

Lemma 1: Any fractional differential equation of order $n\beta$, for $n \in \mathbb{Z}^+$ and $0 < \beta \leq 1$, with functions possessing values in $\mathbb{R}^d$ can be converted into a system of fractional differential equations of order β with values in $\mathbb{R}^{nd}$.

Proof 1: In order to demonstrate this result, we should first take the scaler case that takes place whenever $d = 1$, and then we will consider the remaining case that is hold when $d > 1$. For this reason, we should note that the general forme of the fractional differential equation of fractional order $n\beta$ in its scaler case can be represented by :

$$\begin{aligned} D^{n\beta}y(t) =& G\Big(t, y(t), D^{\beta}y(t), \\ & D^{2\beta}y(t), \ldots, D^{(n-1)\beta}y(t)\Big), \end{aligned} \tag{21}$$

where G is a continuous function defined on the subset $I \times \mathbb{R} \times \ldots \times \mathbb{R}$, so that it takes values in $\mathbb{R}$ for a given interval I. Then, we define:

$$\overrightarrow{\Psi}(t, r_0, \ldots, r_{n-1}) = (r_1, \ldots, r_{n-1}, G(t, r_0, \ldots, r_{n-1})),$$

as continuous function defined on $I \times \mathbb{R} \times \ldots \times \mathbb{R}$ as G, but it takes the values in $\mathbb{R}^n$.
In this regard, we consider the following equation:

$$D^{\beta}\overrightarrow{Y}(t) = \overrightarrow{\Psi}\left(t, \overrightarrow{Y}(t)\right) \quad \text{for } t \in I. \tag{22}$$

Now, we want to show that $x : I \to \mathbb{R}$ is a solution of equation (21) if the function :

$$\begin{aligned} \overrightarrow{X} : I &\longrightarrow \mathbb{R}^n \\ t &\longrightarrow \left(x(t), D^{\beta}x(t), \ldots, D^{(n-1)\beta}x(t)\right), \end{aligned}$$

is a solution of (22). To this end, we suppose that x a solution of equation (21) such that $\overrightarrow{X}$ is defined above. Then we have :

$$\begin{aligned} D^{\beta}\overrightarrow{X}(t) &= \begin{pmatrix} D^{\beta}x(t) \\ \vdots \\ D^{(n-1)\beta}x(t) \\ D^{n\beta}x(t) \end{pmatrix} \\ &= \begin{pmatrix} D^{\beta}x(t) \\ \vdots \\ D^{(n-1)\beta}x(t) \\ G(t, x(t), D^{\beta}x(t), \ldots, D^{(n-1)\beta}x(t)) \end{pmatrix} \\ &= \overrightarrow{\Psi}\left(t, \overrightarrow{X}(t)\right). \end{aligned} \tag{23}$$

Herein, the converse of the above discussion is similar. Now, for the case of $d > 1$, one can re-read the above proof again, and substituting each occurrence of $\mathbb{R}$ by $\mathbb{R}^d$ to get the desired result.

In order to apply the reduction Lemma 1, we will rewrite the system (20) as follows:

$$\begin{cases} D^{2\beta}\psi_1(r) = -\Big(\frac{1}{r}D^{\beta}\psi_1(r) \\ \qquad +\mathcal{C}_{11}\psi_1(r) + \mathcal{C}_{12}\psi_2(r)\Big), \\ D^{2\beta}\psi_2(r) = -\Big(\frac{1}{r}D^{\beta}\psi_2(r) \\ \qquad +\mathcal{C}_{21}\psi_1(r) + \mathcal{C}_{22}\psi_2(r)\Big), \\ \psi_i(a) = \mathcal{A}_i, \quad D^{\beta}\psi_i(a) = \mathcal{B}_i, \ r \in I, \ i = 1, 2, \end{cases} \tag{24}$$

Now, we put

$$\begin{aligned} & f_1\left(r, \psi_1(r), D^{\beta}\psi_1(r), \psi_2(r), D^{\beta}\psi_2(r)\right) \\ & \qquad = -\left(\frac{1}{r}D^{\beta}\psi_1(r) + \mathcal{C}_{11}\psi_1(r) + \mathcal{C}_{12}\psi_2(r)\right), \\ & f_2\left(r, \psi_1(r), D^{\beta}\psi_1(r), \psi_2(r), D^{\beta}\psi_2(r)\right) \\ & \qquad = -\left(\frac{1}{r}D^{\beta}\psi_2(r) + \mathcal{C}_{21}\psi_1(r) + \mathcal{C}_{22}\psi_2(r)\right). \end{aligned} \tag{25}$$

Then system (24) becomes

$$\begin{cases} D^{2\beta}\psi_1(r) = f_1\Big(r, \psi_1(r), D^{\beta}\psi_1(r), \psi_2(r), D^{\beta}\psi_2(r)\Big), \\ D^{2\beta}\psi_2(r) = f_2\Big(r, \psi_1(r), D^{\beta}\psi_1(r), \psi_2(r), D^{\beta}\psi_2(r)\Big), \\ \psi_i(a) = \mathcal{A}_i, \quad D^{\beta}\psi_i(a) = \mathcal{B}_i, \ r \in I, \ i = 1, 2. \end{cases} \tag{26}$$

At this point, we convert the previous system, which is of order 2β into a fractional system of order β. For this purpose, we assume:

$$v_i(r) = D^{\beta}\psi_i(r), \forall i = 1, 2.$$

According to this assumption, the system (26) transforms into the following form:

$$\begin{aligned}
&D^{\beta}\psi_1(r) = v_1(r) = g_1\left(r, \psi_1(r), v_1(r), \psi_2(r), v_2(r)\right)\\
&D^{\beta}v_1(r) = D^{2\beta}\psi_1(r) = f_1\left(r, \psi_1(r), v_1(r), \psi_2(r), v_2(r)\right)\\
&D^{\beta}\psi_2(r) = v_2(r) = g_2\left(r, \psi_1(r), v_1(r), \psi_2(r), v_2(r)\right)\\
&D^{\beta}v_2(r) = D^{2\beta}\psi_2(r) = f_2\left(r, \psi_1(r), v_1(r), \psi_2(r), v_2(r)\right)\\
&\psi_i(a) = \mathcal{A}_i, \quad v_i(a) = \mathcal{B}_i,\ i = 1, 2.
\end{aligned} \tag{27}$$

In the following, we intend also to consider the cylindrical radius r as a time domain. Thus, in order to solve the system (27) via MFEM, we divide the interval $I = [a, b]$ as $a = r_0 < r_1 = r_0 + h < r_2 = r_0 + 2h < \cdots < r_n = r_0 + nh = b$ such that $r_i = r_0 + ih$ and $h = \frac{b-a}{m}$, for $i = 1, 2$.

For the convenience, we denote respectively

$$g_i(r, \psi_1(r), v_1(r), \psi_2(r), v_2(r))$$

and

$$f_i(r, \phi_1(r, \psi_1(r), v_1(r), \psi_2(r), v_2(r))$$

by g_i and f_i, for all $i = 1, 2$. Now, by using the MFEM's main formula (4), we can derive:

$$\begin{cases}
\psi_1(r_{i+1}) = \psi_1(r_i) + \dfrac{h^{\beta}}{\Gamma(\beta+1)} g_1\Bigg(r_i + \dfrac{h^{\beta}}{2\Gamma(\beta+1)},\\
\qquad \psi_1(r_i) + \dfrac{h^{\beta}}{2\Gamma(\beta+1)} g_1, v_1(r_i)\\
\qquad + \dfrac{h^{\beta}}{2\Gamma(\beta+1)} f_1, \quad \psi_2(r_i) + \dfrac{h^{\beta}}{2\Gamma(\beta+1)} g_2,\\
\qquad v_2(r_i) + \dfrac{h^{\beta}}{2\Gamma(\beta+1)} f_2\Bigg),
\end{cases} \tag{28}$$

$$\begin{cases}
v_1(r_{i+1}) = v_1(r_i) + \dfrac{h^{\beta}}{\Gamma(\beta+1)} f_1\Bigg(r_i + \dfrac{h^{\beta}}{2\Gamma(\beta+1)},\\
\qquad \psi_1(r_i) + \dfrac{h^{\beta}}{2\Gamma(\beta+1)} g_1, v_1(r_i)\\
\qquad + \dfrac{h^{\beta}}{2\Gamma(\beta+1)} f_1, \quad \psi_2(r_i) + \dfrac{h^{\beta}}{2\Gamma(\beta+1)} g_2,\\
\qquad v_2(r_i) + \dfrac{h^{\beta}}{2\Gamma(\beta+1)} f_2\Bigg),
\end{cases}$$

$$\begin{cases}
\psi_2(r_{i+1}) = \psi_2(r_i) + \dfrac{h^{\beta}}{\Gamma(\beta+1)} g_2\Bigg(r_i + \dfrac{h^{\beta}}{2\Gamma(\beta+1)},\\
\qquad \psi_1(r_i) + \dfrac{h^{\beta}}{2\Gamma(\beta+1)} g_1, v_1(r_i)\\
\qquad + \dfrac{h^{\beta}}{2\Gamma(\beta+1)} f_1, \quad \psi_2(r_i) + \dfrac{h^{\beta}}{2\Gamma(\beta+1)} g_2,\\
\qquad v_2(r_i) + \dfrac{h^{\beta}}{2\Gamma(\beta+1)} f_2\Bigg),
\end{cases}$$

$$\begin{cases}
v_2(r_{i+1}) = v_2(r_i) + \dfrac{h^{\beta}}{\Gamma(\beta+1)} f_2\Bigg(r_i + \dfrac{h^{\beta}}{2\Gamma(\beta+1)},\\
\qquad \psi_1(r_i) + \dfrac{h^{\beta}}{2\Gamma(\beta+1)} g_1, v_1(r_i)\\
\qquad + \dfrac{h^{\beta}}{2\Gamma(\beta+1)} f_1, \quad \psi_2(r_i) + \dfrac{h^{\beta}}{2\Gamma(\beta+1)} g_2,\\
\qquad v_2(r_i) + \dfrac{h^{\beta}}{2\Gamma(\beta+1)} f_2\Bigg).
\end{cases}$$

for $i = 0, 1$. The above system represents an approximate solution of system (27) and hence $(\psi_1(t), \psi_2(t))$ is then the defined solution of system (20).

IV. Results and Discussion

In this section, we present the parameter values in Table I and Table II, which are derived from the reference [10], such that these data are used to propose a comparison between the exact solution of the integer–order two–energy groups neutron diffusion model in cylindrical reactors by applying LTM and for a numerical solution generated by reduction process using MFEM.

TABLE I. Data of the Two–Energy Groups Neutron Diffusion System

Fast Energy Group	**Thermal Energy Group**
$\sum_{f_1} = 0.0010484cm^{-1}$	$\sum_{f_2} = 0.05063cm^{-1}$
$\sum_{S11} = 0.62568cm^{-1}$	$\sum_{S22} = 2.443838cm^{-1}$
$\sum_{\gamma_1} = 0.0010046cm^{-1}$	$\sum_{\gamma_2} = 0.025788cm^{-1}$
$\sum_{S12} = 0.029227cm^{-1}$	$\sum_{S21} = 0.00000cm^{-1}$
$v_1 = 2.5$	$v_2 = 2.5$
$\chi_1 = 1.0$	$\chi_2 = 0.0$

TABLE II. The Coefficient Values $\mathcal{C}_{ij}$ That Were Determined Using (6) and (7)

$\mathcal{C}_{ij}$	$i = 1$	$j = 2$
$i = 1$	-0.0564834	0.220978
$j = 2$	0.249474	-0.577793

The integer-order two–energy groups of neutron diffusion system related to the cylindrical reactor can be described as follows:

$$\begin{aligned} r\psi_1''(r) + \psi_1'(r) + r\mathcal{C}_{11}\psi_1(r) + r\mathcal{C}_{12}\psi_2(r) = 0, \\ r\psi_2''(r) + \psi_2'(r) + r\mathcal{C}_{21}\psi_1(r) + r\mathcal{C}_{22}\psi_2(r) = 0, \end{aligned} \tag{29}$$

with initial conditions

$$\begin{aligned} \psi_1(0.0001) = 2.766976, \psi_2(0.0001) = 1, \\ \psi_1'(0.0001) = 1, \psi_2'(0.0001) = 0. \end{aligned} \tag{30}$$

With the help of using the proposed scheme in Subsection III-A, we can obtain the solution of the above system in its classical form. On the other hand, the fractional-order of the two–energy groups of neutron reactor diffusion model related to the cylindrical reactor can be outlined in the following form:

$$\begin{aligned} rD^{2\beta}\psi_1(r) + D^{\beta}\psi_1(r) + r\mathcal{C}_{11}\psi_1(r) + r\mathcal{C}_{12}\psi_2(r) = 0, \\ rD^{2\beta}\psi_2(r) + D^{\beta}\psi_2(r) + r\mathcal{C}_{21}\psi_1(r) + r\mathcal{C}_{22}\psi_2(r) = 0, \end{aligned} \tag{31}$$

according to the initial conditions

$$\begin{aligned} \psi_1(0.0001) = 2.766976, \psi_2(0.0001) = 1, \\ D^{\beta}\psi_1(0.0001) = 1, D^{\beta}\psi_2(0.0001) = 0. \end{aligned} \tag{32}$$

For dealing with this system, we apply the process mentioned in the Subsection III-B that uses MFEM. At this step, to verify that the procedure developed by the Lemma 1 is valid, we make a numerical comparison between the LTM's solution of system (29)-(30) and the MFEM's solution of system (31)-(32) in Fig. 1. Given this figure, it is evident that the two solutions completely coincide, meaning that the neutron reactor diffusion model's two–energy groups can be perfectly simulated when $\beta = 1$.

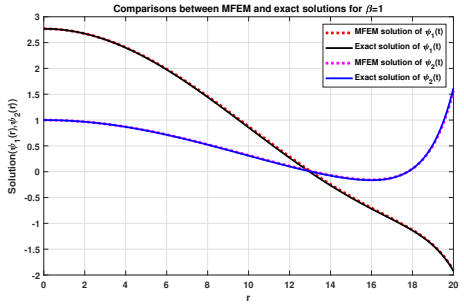

Fig. 1. A graphical comparison: LTM's solution for (29)-(30) & the MFEM's solution for (31)-(32).

In Fig. 2, we present a comparison for $\psi_1(r)$, between the exact solution when $\beta = 1$, and MFEM's solutions according to different values of β.

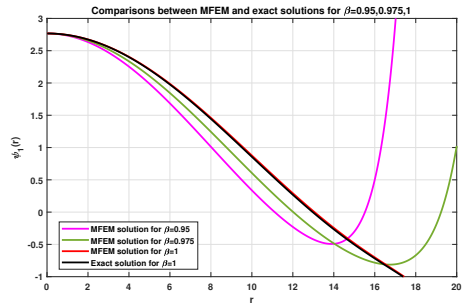

Fig. 2. A graphical comparison between the exact solution of $\psi_1(r)$ and MFEM's numerical solutions for according to $\beta = 0.95, 0.975, 1$ and the exact solution.

In Fig. 3, we depict a numerical comparison for $\psi_2(r)$, which is performed between several MFEM's solutions for different fractional-order values ($\beta = 0.95, 0.975, 1$), and the exact solution.

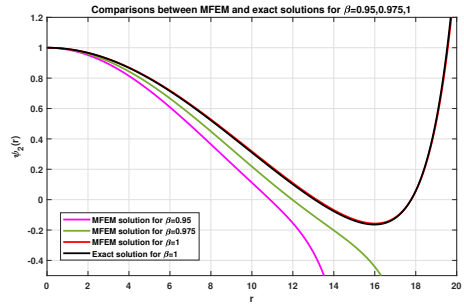

Fig. 3. A graphical comparison between the exact solution of $\psi_2(r)$ and MFEM's numerical solutions for according to $\beta = 0.95, 0.975, 1$ and the exact solution.

In Figs. 4 and 5, respectively, we plot various MFEM's solutions of such a system for different values of $\beta = 0.95, 0.975, 1$, to gain additional clarity into the numerical solutions $(\psi_1(r), \psi_2(r))$.

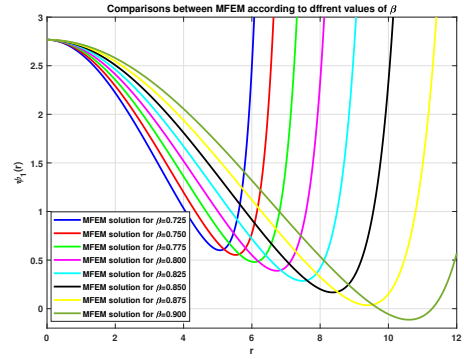

Fig. 4. MFEM's solutions for $\psi_1(r)$ according to different fractional-order values.

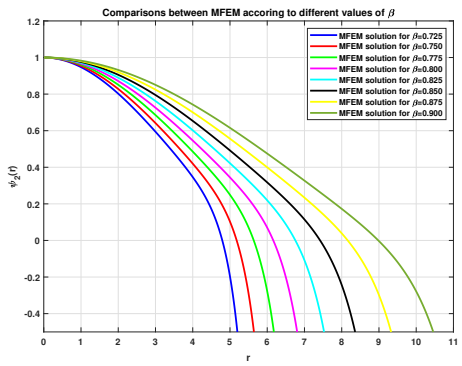

Fig. 5. MFEM's solutions for $\psi_2(r)$ according to different fractional-order values.

The obtained flux values in the proposed fractional-order two–energy groups neutron diffusion model in cylindrical reactors hold significant importance in understanding the behavior of neutrons within the reactor core. Here's a more detailed discussion to aid in the interpretation of the results:

- **Fractional-order dynamics:** We find that the fractional-order neutron diffusion model can allow for a more nuanced representation of neutron transport phenomena. The traditional integer-order model assumes integer values for derivatives, whereas the fractional-order model introduces non-integer orders, capturing more accurately the memory effects and long-range correlations in neutron transport.
- **Sensitivity to memory effects:** The fractional-order derivatives account for memory effects, enabling the model to incorporate information from previous time steps with varying degrees of influence. From this point of view, we find that the neutrons of the proposed model retain a memory of past interactions, and the impact of historical events on the current neutron flux is significant.
- **Impact on reactor stability and performance:** The fractional-order dynamics affect the parameters of the model that are essential for the reactor's behavior, which makes the neutron flux values directly influence reactor stability and performance.
- **Comparison with the traditional model:** When we compare the fractional- and integer-order neutron diffusion models, we can observe that we can identify several differences in flux distributions that can be generated via the fractional-order model, which is regarded as a modification of the traditional approach.

V. Conclusion

In this work, we have proposed solutions for two versions of the two–energy groups neutron diffusion model in cylindrical geometries. The integer-order version has been resolved using the Laplace transform method (LTM). On the other hand, the fractional-order system has been dealt with by applying the modified fractional Euler method (MFEM). A comparison was made via graphical representations between the exact solution of the classical model by applying LTM and the numerical solution of the fractional reduction system using MFEM, and the results were favorable. For a comprehensive evaluation of the considered model, we list the following future directions for this work:

- We believe that incorporating the calculation of the multiplication factor "k_{eff}" for evaluating the neutron diffusion model should be studied, and this direction will be left to the future for further consideration.
- We also believe that the validation of the model against experimental data or benchmark problems is critical for establishing its reliability.
- The neutron flux values are closely linked to reactor safety. Anomalies in flux distributions or unexpected behaviors revealed by the fractional-order model can have safety implications. Understanding these implications is crucial for reactor design, operation, and safety analysis.
- For detailed analyses or when a high level of accuracy is crucial, a multi-energy group model may be more appropriate, despite its increased complexity. In fact, the practical usefulness of multi-energy group analysis stem from its superior accuracy, regulatory compliance, and critical role in advancing nuclear technology. While two-energy group methods offer valuable simplifications for initial studies and education, the detailed insights provided by multi-energy group methods are indispensable for modern reactor design, safety, and innovation. In addition, the integrating multi-energy group analysis enhances the robustness and reliability of nuclear engineering practices, making it an essential tool in the field. However, studying multi–energy groups of a neutron diffusion system for cylindrical reactors will be left to the future consideration.

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