Revolutionizing Numerical Approximations: A Novel Higher-Order Implicit Method vs. Runge-Kutta for Initial Value Problems

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Abstract—**This work is dedicated to advancing the approximation of initial value problems through the introduction of an innovative and superior method inspired by Taylor's approach. Specifically, we present an enhanced variant achieved by accelerating the expansion of the Obreschkoff formula. This results in a higher-order implicit corrected method that outperforms Rung–Kutta's (RK) method in terms of accuracy. We derive an error bound for the Obreschkoff higher-order method, showcasing its stability, convergence, and greater efficiency than the conventional RK method. To substantiate our claims, numerical experiments are provided, highlighting the exceptional efficacy of our proposed method over the traditional RK method.**

Keywords—Obreschkoff Method; Rung-Kutta Method; ODE; Darboux's formula; Approximations

I. Introduction

Initial value problems (I.V.Ps) form the crux of modeling dynamic systems across scientific and engineering disciplines, compelling the quest for precise and efficient solutions through numerical approximation methods [1]–[13]. In the landscape of established methods, the Euler method, despite its simplicity, serves as a foundational approach, providing a baseline for numerical integration. Modifications to the Euler method, in pursuit of enhanced accuracy and efficiency, have yielded a spectrum of techniques. To see more studies about the IVPs and their generalizations, applications, and more, the reader may refer to the references [14]–[33].

The Taylor method, distinguished by its systematic series expansion, stands out for its potential to deliver more intricate and accurate approximations. However, its demand for increased computational resources prompts exploration into alternative methods such as the Midpoint method. This member of the Runge-Kutta family strikes a balance between accuracy and computational cost, making it particularly relevant for problems with moderate complexity.

The most recognized method is the Runge-Kutta method, which approximates the solution of ordinary differential equations (ODEs). It is an iterative method that computes approximate solutions by taking weighted averages of several intermediate steps. The method is known for its accuracy and versatility, with different orders of the method providing varying levels of precision. The basic idea involves estimating the slope of the solution curve at various points within a given interval and using these estimates to iteratively refine the solution. This method is widely used in scientific and engineering fields where analytical solutions to ODEs are difficult or impossible to obtain.

There is a huge backlog of good ingenious recent works concerning the Runge-Kutta and Taylor methods, and their consequences or other related methods. However, we tried our best to attract the reader's attention to the following recent works and the references therein [34], [35]–[54]. For more about the Taylor method and its consequences or other related methods.

In the realm of approximating initial value problems (I.V.Ps), our work introduces a new analytical method that stands out as a significant departure from traditional approaches. Unlike conventional methods, our approach is not merely an adaptation or incorporation of existing techniques, but rather a pioneering and original construction tailored to address the intricacies of I.V.Ps in ordinary differential equations (ODEs). This analytical method, which encompasses a broader spectrum of solutions, transcends the confines of established methodologies, presenting a novel framework for tackling the challenges associated with I.V.Ps.

The core of our analytical method involves an accelerated expansion of the Obreschkoff formula, resulting in a higher-order implicit corrected method that redefines the landscape of approximating solutions to I.V.Ps. Notably, our approach does not treat the Taylor method as a foundational basis but rather positions it as a special case within our more encompassing framework. Through rigorous theoretical analysis and derivation of error bounds, we establish the stability, convergence, and efficiency of our analytical method. We unveil a groundbreaking approach designed to significantly enhance the accuracy and efficiency of approximating initial value problems governed by ordinary differential equations (ODEs).

Numerical experiments conducted to validate the capabilities of our method further amplify its significance. These experiments not only demonstrate the robustness and accuracy of our analytical method but also underscore its superiority when compared to conventional approaches, including the traditional Taylor method. In summary, our analytical method for approximating I.V.Ps in ODEs represents a pioneering contribution, offering a fresh perspective that extends beyond the limitations of existing techniques and provides a more comprehensive and effective solution for a diverse range of mathematical contexts.

Throughout this work, let I be a real interval, and let $a, b \in I^{\circ}$ (the interior of I) with $a < b$. We denote by $P_n(t)$ a polynomial of degree n defined on $[a, b]$. Furthermore, let $\mathcal{P}_n(I)$ be the class of polynomials of degree n defined on an interval $I \subseteq \mathbb{R}$. Let $f(x)$ be analytic at all points of the interval $[a, x]$, and let $\phi(t) \in \mathcal{P}_n$. If $t \in [0, 1]$, by differentiation [55, p. 125], we obtain

$$\begin{aligned}\frac{d}{dt}&\sum_{k=1}^{n}(-1)^k(x-a)^k\phi^{(n-k)}(t)f^{(k)}(a+t(x-a))\\&=-(x-a)\phi^{(n)}(t)f'(a+t(x-a))\\&\quad+(-1)^n(x-a)^{n+1}\phi(t)f^{(n+1)}(a+t(x-a)).\end{aligned}$$

Since $\phi^{(n)}(t) = \phi^{(n)}(0) =$ constant, we integrate from 0 to 1 with respect to t and obtain

$$\begin{aligned}\phi^{(n)}(0)\left[f(x)-f(a)\right]&=\sum_{k=1}^{m}(-1)^{k-1}(x-a)^k\\&\quad\times\left\{\phi^{(n-k)}(1)f^{(k)}(x)-\phi^{(n-k)}(0)f^{(k)}(a)\right\}\\&\quad+(-1)^n(x-a)^{n+1}\int_0^1\phi(t)f^{(n+1)}(a+t(x-a))dt.\end{aligned}\tag{1}$$

This result is known as *Darboux's formula* [56].

Suppose that $f(x)$ is defined on an interval $I \subset \mathbb{R}$ and that $f^{(n)}(x)$ is absolutely continuous on I. Let $P_n(t)$ be a polynomial of degree n with the coefficient of the term t^n equal to a_n, and let $a \in I$. Then,

$$\begin{aligned}f(x)=f(a)&+\sum_{k=1}^{n}\frac{(-1)^{k-1}}{n!a_n}\\&\times\left[P_n^{(n-k)}(x)f(x)-P_n^{(n-k)}(a)f(a)\right]\\&+\frac{(-1)^n}{n!a_n}\int_a^x P_n(t)f^{(n+1)}(t)\,dt.\end{aligned}\tag{2}$$

This is a modified version of (1), which was proved in [**?**] (see also [57]).

The concept of a harmonic sequence of polynomials, also known as Appell polynomials, has been widely used in numerical integration and expansion theory of real functions. We recall that a sequence of polynomials $\{P_k(t,\cdot)\}_{k=0}^{\infty}$ satisfies the Appell condition (see [58]) if

$$\frac{\partial}{\partial t}P_k(t,\cdot)=P_{k-1}(t,\cdot),\quad \forall k\geq 1,$$

with the initial condition

$$P_0(t,\cdot)=1.$$

This holds for all well-defined ordered pairs $(t,\cdot)$. A slightly different definition was considered in [59].

In fact, Matić et al. in [59] established the following generalization of the Taylor formula using the concept of a harmonic sequence of polynomials:

Theorem 1. Let $\{Q_n\} \subset \mathcal{P}_n(I)$ be a harmonic sequence of polynomials, i.e., $Q_n' = Q_{n-1}$ for all $n \in \mathbb{N}$, with $Q_0 = 1$. Let $I \subset \mathbb{R}$ be a closed interval such that $a \in I$. If $f : I \to \mathbb{R}$ is a function such that, for some $n \in \mathbb{N}$, $f^{(n)}$ is absolutely continuous, then

$$\begin{aligned}f(x)=f(a)&+\sum_{k=1}^{n}(-1)^{k+1}\left[Q_k(x)f^{(k)}(x)-Q_k(a)f^{(k)}(a)\right]\\&+(-1)^n\int_a^x Q_n(t)f^{(n+1)}(t)\,dt,\end{aligned}\tag{3}$$

for any $x \in I$.

Clearly, by setting $H_n(t) = \frac{(t-x)^k}{k!}$, we recover the Taylor formula. Constructing the sequence of polynomials $H_k(t)$ can be described as follows:

$$1,\frac{1}{n!a_n}P_n^{(n-1)}(t),\frac{1}{n!a_n}P_n^{(n-2)}(t),\ldots,\frac{1}{n!a_n}P_n'(t),\frac{1}{n!a_n}P_n(t)$$

in which $a_n \neq 0$. Equivalently, we write

$$H_k(t)=\frac{1}{n!a_n}P_n^{(n-k)}(t),\qquad 1\leq k\leq n,\tag{4}$$

with $a_n \neq 0$.

Lemma 1. The sequence $H_k(t) \in \mathcal{P}_n(I)$ $(1 \le k \le n)$, defined in (12), forms a harmonic sequence of polynomials.

Proof. Clearly, since

$$H_k'(t) = \frac{1}{n!a_n}\frac{d}{dt}P_n^{(n-k)}(t) = \frac{1}{n!a_n}P_n^{(n-k+1)}(t) = H_{k-1}(t),$$

for all $k = 1, 2, \ldots, n$. Also, we have

$$H_0(t) = \frac{1}{n!a_n}P_n^{(n)}(t) = 1,$$

which proves that $H_k(t)$ forms a harmonic sequence of polynomials.

Therefore, we have shown that the expansion by Matić et al. is a special case of the original Darboux formula. Moreover, it is possible to expand f using any general sequence of polynomials.

In the next lemma, we construct a generalized Euler–Maclaurin formula for a general sequence of polynomials.

Lemma 2. Let $\{P_n\} \subset \mathcal{P}_n(I)$ be any sequence of polynomials. Let $f : I \to \mathbb{R}$ be such that $f^{(n)}$ is absolutely continuous on I, and assume that $P_n(t)f^{(n)}(t)$ is integrable. Then, we have the representation

$$\int_a^b f(u)\,du = Q_n(f; P_n) + R_n(f; P_n), \tag{5}$$

where

$$Q_n(f; P_n) := \sum_{k=1}^{n} \frac{(-1)^k}{n!a_n} \times \left[P_n^{(n-k)}(a)f^{(k-1)}(a) - P_n^{(n-k)}(b)f^{(k-1)}(b)\right] \tag{6}$$

and

$$R_n(f; P_n) := \frac{(-1)^n}{n!a_n}\int_a^b P_n(t)f^{(n)}(t)\,dt. \tag{7}$$

Proof. Starting with the Darboux expansion for f along $[a, b]$,

$$\begin{aligned} f(x) = f(y) + \sum_{k=1}^{n} \frac{(-1)^k}{n!a_n} &\times \left[P_n^{(n-k)}(y)f^{(k)}(y) - P_n^{(n-k)}(x)f^{(k)}(x)\right] \\ &+ \frac{(-1)^n}{n!a_n}\int_y^x P_n(t)f^{(n+1)}(t)\,dt. \end{aligned} \tag{8}$$

If we set $x = a$, $y = b$, and replace $f(t)$ by $\int_a^t f(u)\,du$ in (8), we obtain the desired representation (5), which completes the proof.

Theorem 2. Let $\{H_n\} \subset \mathcal{P}_n(I)$ be a harmonic sequence of polynomials, i.e., $H_n' = H_{n-1}$ for all $n \in \mathbb{N}$ with $H_0 = 1$. Let $I \subset \mathbb{R}$ be a closed interval such that $a \in I^\circ$ is fixed. If $f : I \to \mathbb{R}$ is real analytic on I°, then

$$f(x) = f(a) + \sum_{k=1}^{\infty}(-1)^{k+1}\left[H_k(x)f^{(k)}(x) - H_k(a)f^{(k)}(a)\right] \tag{9}$$

for any $x \in I$.

Proof. Since $f^{(n+1)}$ is continuous on I, from (3), we have

$$\begin{aligned} |R_n(x)| &= \left|(-1)^n\int_a^x H_n(t)f^{(n+1)}(t)\,dt\right| \\ &\le \int_a^x |H_n(t)|\left|f^{(n+1)}(t)\right| dt \\ &\le \sup_{t\in[a,x]}\left|f^{(n+1)}(t)\right|\int_a^x |H_n(t)|\,dt. \end{aligned}$$

Now, since H_n is a harmonic sequence of polynomials, there exists $n_0 \in \mathbb{N}$ such that

$$|H_n(t)| \le \frac{1}{n!}|x - t|^n, \quad \forall n \ge n_0.$$

Assuming the radius of convergence is $|x - t| = \rho$, we obtain

$$\begin{aligned} |R_n(x)| &\le \sup_{t\in[a,x]}\left|f^{(n+1)}(t)\right|\frac{|x-t|^{n+1}}{(n+1)!} \\ &= M \cdot \frac{\rho^{n+1}}{(n+1)!} \longrightarrow 0, \quad \text{as} \quad n \to \infty. \end{aligned}$$

This gives the expansion in (9), completing the proof.

A direct consequence of the above theorem is Taylor's theorem. Specifically, if one chooses

$$H_n(t) = \frac{(t-x)^n}{n!},$$

then (9) reduces to the well-known Taylor expansion of the real function f near a point a.

In view of Lemma 2, we can generalize Theorem 2 for any sequence of polynomials as follows:

Corollary 1. Let $\{P_{n-k}\} \subset \mathcal{P}_n(I)$ $(1 \le k \le n)$ be any sequence of polynomials. Let $I \subset \mathbb{R}$ be a closed interval such that $a \in I^\circ$ is fixed. If $f : I \to \mathbb{R}$ is real analytic on I°, then

$$f(x) = f(a) + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{n!a_n} \times \left[P_n^{(n-k)}(x)f^{(k)}(x) - P_n^{(n-k)}(a)f^{(k)}(a)\right] \tag{10}$$

for any $x \in I$, where $a_n \neq 0$ is the leading coefficient of P_{n-k}.

On the other hand, in 1949, Hummel and Seebeck [60] independently established a generalization of Taylor's expansion, where they developed a power series expansion that accelerates convergence twice as rapidly as the Taylor expansion. In fact, they proved that

$$\begin{aligned} f(x) = f(y) &+ \sum_{k=1}^{m} \frac{\binom{m}{k}}{\binom{m+n}{k}} \frac{f^{(k)}(y)}{k!}(x-y)^k \\ &+ \sum_{k=1}^{n} (-1)^{k+1} \frac{\binom{n}{k}}{\binom{m+n}{k}} \frac{f^{(k)}(x)}{k!}(x-y)^k \\ &+ \frac{(-1)^n}{(m+n)!}\int_y^x (x-t)^m (t-y)^n f^{(m+n+1)}(t)dt. \end{aligned} \tag{11}$$

This expansion is valid for all real-valued functions $f \in C^{m+n+1}(I)$, where $m, n = 0, 1, 2, \dots$, for some interval I with $y \in I$. Moreover, Hummel and Seebeck proved that

$$\begin{aligned} f(x) = f(y) &+ \sum_{k=1}^{m} \frac{\binom{m}{k}}{\binom{m+n}{k}} \frac{f^{(k)}(y)}{k!}(x-y)^k \\ &+ \sum_{k=1}^{n} (-1)^{k+1} \frac{\binom{n}{k}}{\binom{m+n}{k}} \frac{f^{(k)}(x)}{k!}(x-y)^k \\ &+ (-1)^n \frac{m!n!(x-y)^{m+n+1}}{(m+n)!(m+n+1)!} f^{(m+n+1)}(\xi(x)) \end{aligned}$$

for some $\xi(x) \in (a, x)$.

Historically, the formula (11) was established ten years earlier by Obreschkoff [61] in 1940. Due to the Second World War, such an overlap in work may have occurred. The works of Obreschkoff and Hummel–Seebeck represent a special case of a more general formula established by Darboux himself in 1876, as mentioned in [55, p. 125] and [62]. In the special case where $n = 0$ in (11), we obtain the celebrated Taylor polynomials along with their error term. As described by Hummel and Seebeck, the case where $n = m$ is particularly interesting because it exhibits a faster rate of convergence compared to the Taylor expansion. This can be observed by analyzing the remainder term in (11) (see [60]).

Remark 1. Setting $P_n(t) = (t - x)^n$ in (2) yields the well-known Taylor expansion formula. While it has been emphasized that the Hummel–Seebeck and Obreschkoff formulas share an identical nature, an intriguing observation arises when we introduce a subtle transformation. Specifically, substituting $P_n(t)$ with

$$P_{n+m}(t) = (t - x)^m (t - a)^n$$

in the Darboux formula (2) remarkably leads to an equivalent result. This substitution not only highlights the robustness of the Darboux formula but also reveals a deeper connection between these mathematical representations. The seamless interchangeability of these components underscores the intricate relationships embedded within these formulations, offering a richer perspective on their interplay and structural coherence.

II. The Obreschkoff Higher-Order Method

This method aims to obtain an approximation for the well-posed initial-value problem

$$\frac{dy}{dt} = f(t, y), \qquad a \le t \le b, \qquad y(a) = \alpha. \tag{12}$$

Suppose the solution $y(t)$ to the initial-value problem has $(m+n+1)$ continuous derivatives. Expanding $y(t)$ in terms of its $(m+n)$-th Obreschkoff polynomial about t_i and evaluating at t_{i+1}, we obtain

$$\begin{aligned} y(t_{i+1}) = y(t_i) &+ \sum_{k=1}^{m} \frac{\binom{m}{k}}{\binom{m+n}{k}} \frac{y^{(k)}(t_i)}{k!}(t_{i+1}-t_i)^k \\ &+ \sum_{k=1}^{n} (-1)^{k+1} \frac{\binom{n}{k}}{\binom{m+n}{k}} \frac{y^{(k)}(t_{i+1})}{k!}(t_{i+1}-t_i)^k \\ &+ (-1)^n \frac{m!n!(t_{i+1}-t_i)^{m+n+1}}{(m+n)!(m+n+1)!} y^{(m+n+1)}(\xi_i), \end{aligned}$$

for some $\xi_i \in (t_i, t_{i+1})$.

We begin by establishing the condition that the distribution of mesh points is uniform across the interval $[a, b]$. This requirement is ensured by selecting a positive integer N, from which the mesh points are defined as

$$t_i = a + ih, \qquad \text{for each } i = 0, 1, 2, \dots, N.$$

The step size, or uniform spacing between the points, is given by

$$h = \frac{b - a}{N} = t_{i+1} - t_i.$$

Suppose that the unique solution to (12) has $(m+n+1)$ continuous derivatives on $[a, b]$. Then, for each $i = 0, 1, 2, \dots, N-1$, we have

$$\begin{aligned} y(t_{i+1}) = y(t_i) &+ \sum_{k=1}^{m} \frac{\binom{m}{k}}{\binom{m+n}{k}} \frac{y^{(k)}(t_i)}{k!} h^k \\ &+ \sum_{k=1}^{n} (-1)^{k+1} \frac{\binom{n}{k}}{\binom{m+n}{k}} \frac{y^{(k)}(t_{i+1})}{k!} h^k \\ &+ (-1)^n \frac{m!n!h^{m+n+1}}{(m+n)!(m+n+1)!} y^{(m+n+1)}(\xi_i), \end{aligned}$$

for some $\xi_i \in (t_i, t_{i+1})$. Since $y(t)$ satisfies the differential equation (12), successive differentiation of the solution $y(t)$ gives

$$y'(t) = f(t, y(t)), \ \dots, y^{(k)}(t) = f^{(k-1)}(t, y(t)).$$

Substituting these expressions into (13) yields

$$\begin{aligned}y(t_{i+1}) = y(t_i) &+ \sum_{k=1}^{m} \frac{\binom{m}{k}}{\binom{m+n}{k}} \frac{h^k}{k!} f^{(k-1)}(t_i, y(t_i)) \\ &+ \sum_{k=1}^{n} (-1)^{k+1} \frac{\binom{n}{k}}{\binom{m+n}{k}} \frac{h^k}{k!} f^{(k-1)}(t_{i+1}, y(t_{i+1})) \\ &+ (-1)^n \frac{m!n!h^{m+n+1}}{(m+n)!(m+n+1)!} f^{(m+n)}(\xi_i, y(\xi_i))\end{aligned} \tag{13}$$

The difference-equation method corresponding to (13) is obtained by omitting the remainder term involving ξ_i. This yields

$$\begin{aligned}w_0 &= \alpha, \\ w_{i+1} &= w_i + hB_1^{(m)}(t_i, w_i) + hB_2^{(n)}(t_{i+1}, w_{i+1}),\end{aligned} \tag{14}$$

for each $i = 0, 1, 2, \ldots, N-1$, where

$$B_1^{(m)}(t_i, w_i) := \sum_{k=1}^{m} \frac{\binom{m}{k}}{\binom{m+n}{k}} \frac{h^{k-1}}{k!} f^{(k-1)}(t_i, w_i),$$

and

$$B_2^{(n)}(t_{i+1}, w_{i+1}) := \sum_{k=1}^{n} (-1)^{k+1} \frac{\binom{n}{k} h^{k-1}}{\binom{m+n}{k} k!} f^{(k-1)}(t_{i+1}, w_{i+1}).$$

It is worth noting that if $n = 1$, then (14) recaptures the well-known general Taylor's method of higher order. In particular, we are interested in the case when $m = n$, in which case (14) simplifies to the difference equation

$$\begin{aligned}w_0 &= \alpha, \\ w_{i+1} &= w_i + h\widetilde{B}_1^{(n)}(t_i, w_i) + h\widetilde{B}_2^{(n)}(t_{i+1}, w_{i+1}),\end{aligned} \tag{15}$$

for each $i = 0, 1, 2, \ldots, N-1$, where

$$\widetilde{B}_1^{(n)}(t_i, w_i) := \sum_{k=1}^{n} \frac{\binom{n}{k}}{\binom{2n}{k}} \frac{h^{k-1}}{k!} f^{(k-1)}(t_i, w_i),$$

and

$$\widetilde{B}_2^{(n)}(t_{i+1}, w_{i+1}) := \sum_{k=1}^{n} (-1)^{k+1} \frac{\binom{n}{k} h^{k-1}}{\binom{2n}{k} k!} f^{(k-1)}(t_{i+1}, w_{i+1}).$$

A. The Initial One-Step Obreschkoff Method

The initial one-step Obreschkoff method unexpectedly exhibits similarities to the widely recognized trapezoidal method [63, p. 351]. However, our methodology deviates from the standard trapezoidal method in its formulation and construction. We now present our approach in the following manner. Suppose that the unique solution to (12) has a third continuous derivative on $[a, b]$. Then, for each $i = 0, 1, 2, \ldots, N-1$, we have

$$\begin{aligned}y(t_{i+1}) = y(t_i) &+ \frac{1}{2}(t_{i+1} - t_i) y'(t_i) \\ &+ \frac{1}{2}(t_{i+1} - t_i) y'(t_{i+1}) - \frac{(t_{i+1} - t_i)^3}{12} y'''(\xi_i),\end{aligned}$$

for some $\xi_i \in (t_i, t_{i+1})$. Since $y(t)$ satisfies the differential equation (12), we can express this as

$$y(t_{i+1}) = y(t_i) + \frac{h}{2} y'(t_i) + \frac{h}{2} y'(t_{i+1}) - \frac{h^3}{12} y'''(\xi_i).$$

Additionally, since $y(t)$ satisfies the differential equation (12), we substitute $y'(t) = f(t, y(t))$ to obtain

$$\begin{aligned}y(t_{i+1}) = y(t_i) &+ \frac{h}{2} f(t_i, y(t_i)) \\ &+ \frac{h}{2} f(t_{i+1}, y(t_{i+1})) - \frac{h^3}{12} y'''(\xi_i).\end{aligned}$$

Obreschkoff–Euler's method, also referred to as the new modified Euler's method, constructs $w_i = y(t_i)$ for each $i = 1, 2, \ldots, N$ by omitting the remainder term. Hence, Obreschkoff–Euler's method is described as follows:

$$\begin{aligned}w_0 &= \alpha, \\ w_{i+1} &= w_i + \frac{h}{2} f(t_i, w_i) + \frac{h}{2} f(t_{i+1}, w_{i+1}),\end{aligned}$$

for each $i = 0, 1, 2, \ldots, N-1$. It is straightforward to show that this method is convergent, stable, and has an order of accuracy equal to 3. Next, we present the difference equation for a special case of (15).

B. Obreschkoff Method for $n = 4$

The Obreschkoff method is a high-order numerical approach for solving initial value problems (IVPs). In this subsection, we derive its formulation for the specific case of $n = 4$. The Obreschkoff method when $n = 4$ is given by the difference equation

$$\begin{aligned}w_0 &= \alpha, \\ w_{i+1} &= w_i + h\widetilde{B}_1^{(4)}(t_i, w_i) + h\widetilde{B}_2^{(4)}(t_{i+1}, w_{i+1}),\end{aligned} \tag{16}$$

for each $i = 0, 1, 2, \ldots, N-1$, where

$$\begin{aligned}\widetilde{B}_1^{(4)}(t_i, w_i) := &\frac{1}{2} f(t_i, w_i) + \frac{3h}{28} f'(t_i, w_i) \\ &+ \frac{h^2}{84} f''(t_i, w_i) + \frac{h^3}{1680} f^{(3)}(t_i, w_i),\end{aligned}$$

and

$$\begin{aligned}\widetilde{B}_2^{(4)}(t_{i+1}, w_{i+1}) := &\frac{1}{2} f(t_{i+1}, w_{i+1}) - \frac{3h}{28} f'(t_{i+1}, w_{i+1}) \\ &+ \frac{h^2}{84} f''(t_{i+1}, w_{i+1}) - \frac{h^3}{1680} f^{(3)}(t_{i+1}, w_{i+1}).\end{aligned}$$

Proposition 1. The Obreschkoff method (16) is of order 8.

Proof. Substituting the exact solution into the Taylor expansion, we obtain after long simplification that

$$\begin{aligned}&y(t_{i+1})-y(t_i)-\frac{h}{2}f(t_i,y_i)-\frac{3h^2}{28}f'(t_i,y_i)-\frac{h^3}{84}f''(t_i,y_i)\\&-\frac{h^4}{1680}f'''(t_i,y_i)-\frac{h}{2}f(t_{i+1},y_{i+1})+\frac{3h^2}{28}f'(t_{i+1},y_{i+1})\\&-\frac{h^3}{84}f''(t_{i+1},y_{i+1})+\frac{h^4}{1680}f'''(t_{i+1},y_{i+1})\\&=O(h^8),\end{aligned}$$

which confirms that (16) is of order 8.

Remark 2. In general, by induction, one can observe that the Obreschkoff method has an order of accuracy $O(h^{2n})$.

III. Convergence and Stability of the General Obreschkoff Method

To prove the convergence and establish an error bound for the general Obreschkoff method (15), we require the following key lemma [63, Lemma 5.8, p. 270].

Lemma 3. If s and t are positive real numbers, and $\{a_i\}_{i=1}^{k}$ is a sequence satisfying $a_0 \ge -t/s$, then

$$a_{i+1}\le \exp\left((1+i)s\right)\left(a_0+\frac{t}{s}\right)-\frac{t}{s}.$$

In the following result, we prove that the Obreschkoff method of order $2n$ is convergent and derive an error bound.

Theorem 3. Suppose that $f^{(k)}$ $(0\le k\le 2n-1)$ are continuous and satisfy the Lipschitz condition with constant L_k on the domain

$$D:=\{(t,y): a\le t\le b, -\infty<y<\infty\}.$$

Further, assume that there exists a constant M such that

$$\left|f^{(2n)}(t,y(t))\right|\le M,\quad \forall t\in[a,b],$$

where $y(t)$ denotes the unique solution to the initial-value problem

$$y'=f(t,y),\qquad a\le t\le b,\qquad y(a)=\alpha.$$

Let $w_0,w_1,\dots,w_N$ be the approximations generated by the Obreschkoff method (15) for some positive integer N. Then, the general Obreschkoff method described in (15) is convergent.

Proof. When $i=0$, the assertion holds trivially since $y(t_0)=w_0=\alpha$. Otherwise, from (15) and for $m=n$, we have

$$\begin{aligned}y(t_{i+1})&=y(t_i)+\sum_{k=1}^{n}\frac{\binom{n}{k}}{\binom{2n}{k}}\\&\times\frac{h^k}{k!}\left[f^{(k-1)}(t_i,y(t_i))+(-1)^{k+1}f^{(k-1)}(t_{i+1},y(t_{i+1}))\right]\\&+(-1)^n\frac{(n!)^2h^{2n+1}}{(2n)!(2n+1)!}f^{(2n)}(\xi_i,y(\xi_i)),\end{aligned}$$

for $i=0,1,\dots,N-1$. Similarly, from the equations in (15), we obtain

$$\begin{aligned}w_{i+1}&=w_i+\sum_{k=1}^{n}\frac{\binom{n}{k}}{\binom{2n}{k}}\\&\times\frac{h^k}{k!}\left[f^{(k-1)}(t_i,w_i)+(-1)^{k+1}f^{(k-1)}(t_{i+1},w_{i+1})\right],\end{aligned}$$

for each $i=0,1,2,\dots,N-1$. Utilizing the notations $y_i=y(t_i)$ and $y_{i+1}=y(t_{i+1})$, we obtain the following by subtracting the two equations:

$$\begin{aligned}y_{i+1}-w_{i+1}&=y_i-w_i\\&+\sum_{k=1}^{n}\frac{\binom{n}{k}}{\binom{2n}{k}}\frac{h^k}{k!}\left[f^{(k-1)}(t_i,y_i)-f^{(k-1)}(t_i,w_i)\right]\\&+\sum_{k=1}^{n}(-1)^{k+1}\frac{\binom{n}{k}}{\binom{2n}{k}}\\&\times\frac{h^k}{k!}\left[f^{(k-1)}(t_{i+1},y_{i+1})-f^{(k-1)}(t_{i+1},w_{i+1})\right]\\&+(-1)^n\frac{(n!)^2h^{2n+1}}{(2n)!(2n+1)!}f^{(2n)}(\xi_i,y(\xi_i)).\end{aligned}$$

Applying the triangle inequality, we obtain

$$\begin{aligned}|y_{i+1}-w_{i+1}|&\le|y_i-w_i|\\&+\sum_{k=1}^{n}\frac{\binom{n}{k}}{\binom{2n}{k}}\frac{h^k}{k!}\left|f^{(k-1)}(t_i,y_i)-f^{(k-1)}(t_i,w_i)\right|\\&+\sum_{k=1}^{n}\frac{\binom{n}{k}}{\binom{2n}{k}}\frac{h^k}{k!}\\&\times\left|f^{(k-1)}(t_{i+1},y_{i+1})-f^{(k-1)}(t_{i+1},w_{i+1})\right|\\&+\frac{(n!)^2h^{2n+1}}{(2n)!(2n+1)!}\left|f^{(2n)}(\xi_i,y(\xi_i))\right|.\end{aligned}$$

Now, the function $f^{(k-1)}$ $(k=1,2,\dots,n)$ satisfies the Lipschitz condition in the second variable with a constant denoted as

$$L:=\max_{1\le k\le 2n-1}\{L_k\},$$

and it holds that

$$\left|f^{(2n)}(t,y(t))\right|\le M.$$

Thus, we obtain

$$\begin{aligned}|y_{i+1}-w_{i+1}|&\le|y_i-w_i|+L\sum_{k=1}^{n}\frac{\binom{n}{k}}{\binom{2n}{k}}\frac{h^k}{k!}|y_i-w_i|\\&+L\sum_{k=1}^{n}\frac{\binom{n}{k}}{\binom{2n}{k}}\frac{h^k}{k!}|y_{i+1}-w_{i+1}|\\&+\frac{(n!)^2h^{2n+1}}{(2n)!(2n+1)!}\left|f^{(2n)}(\xi_i,y(\xi_i))\right|.\end{aligned}$$

For simplicity, let us define the following terms:

$$S_n(L,h) := \left(1 + L\sum_{k=1}^{n} \frac{\binom{n}{k}}{\binom{2n}{k}} \frac{h^k}{k!}\right),$$
$$C_n(L,h) := \left(1 - L\sum_{k=1}^{n} \frac{\binom{n}{k}}{\binom{2n}{k}} \frac{h^k}{k!}\right),$$

and

$$E_n(h) := 2\sum_{k=1}^{n} \frac{\binom{n}{k}}{\binom{2n}{k}} \frac{h^{k-1}}{k!}.$$

Before proceeding further, we note that

$$\begin{aligned}\frac{1}{2}LhE_n(h) &= L\sum_{k=1}^{n} \frac{\binom{n}{k}}{\binom{2n}{k}} \frac{h^k}{k!} \le L\cdot \max_{1\le k\le n}\{h^k\}\cdot \sum_{k=1}^{n} \frac{\binom{n}{k}}{\binom{2n}{k}} \frac{1}{k!}\\ &\approx L\cdot \max_{1\le k\le n}\{h^k\}\cdot 0.6968600167\\ &= K\cdot 0.6968600167,\end{aligned}$$

where we used Robbins' inequality [64], which states that for every positive integer k,

$$\sqrt{2\pi k}\left(\frac{k}{e}\right)^k e^{\frac{1}{12k+1}} \le k! \le \sqrt{2\pi k}\left(\frac{k}{e}\right)^k e^{\frac{1}{12k}}.$$

Considering our ultimate interest in allowing $h \to 0^+$, it is reasonable to assume that

$$\frac{1}{2}LhE_n(h) < 0.6968600167K,$$

where K is some fixed, nonzero positive real number, without any adverse consequences. Consequently, we can infer that

$$\begin{aligned}|y_{i+1} - w_{i+1}| &\le \frac{S_n(L,h)}{C_n(L,h)}\cdot |y_i - w_i|\\ &\quad + \frac{(n!)^2 h^{2n+1}}{(2n)!(2n+1)!C_n(L,h)}\cdot M\\ &= \left(1 + \frac{S_n(L,h) - C_n(L,h)}{C_n(L,h)}\right)\cdot |y_i - w_i|\\ &\quad + \frac{(n!)^2 h^{2n+1}}{(2n)!(2n+1)!C_n(L,h)}\cdot M\\ &= \left(1 + \frac{LhE_n(h)}{C_n(L,h)}\right)\cdot |y_i - w_i|\\ &\quad + \frac{(n!)^2 h^{2n+1}}{(2n)!(2n+1)!C_n(L,h)}\cdot M.\end{aligned}$$

Employing Lemma 3, with

$$s(h) = \frac{LhE_n(h)}{C_n(L,h)}, \qquad t(h) = \frac{(n!)^2 h^{2n+1}}{(2n)!(2n+1)!C_n(L,h)}\cdot M,$$

and

$$a_j = |y_j - w_j|, \quad \text{for each } j = 0,1,2,\dots,N,$$

we observe that

$$\begin{aligned}|y_{i+1} - w_{i+1}| \le \exp\left((i+1)\cdot\frac{LhE_n(h)}{C_n(L,h)}\right)\\ \times\left(|y_0 - w_0| + \frac{t(h)}{s(h)}\right) - \frac{t(h)}{s(h)}.\end{aligned}$$

Since $|y_0 - w_0| = 0$, we obtain

$$\lim_{h\to 0^+}\frac{LhE_n(h)}{C_n(L,h)} = 0, \qquad \text{and} \qquad \lim_{h\to 0^+}\frac{t(h)}{s(h)} = 0.$$

Thus, we have

$$\lim_{h\to 0^+}\max_{1\le i\le N}|y_{i+1} - w_{i+1}| = 0,$$

which implies that w_{i+1} converges to y_{i+1}. Consequently, the Obreschkoff method of order $2n$ is convergent, as required.

Theorem 4. Under the assumptions of Theorem 3, we have

$$|y_{i+1} - w_{i+1}| \le \frac{t(h)}{s(h)}\cdot\left(\exp\left((t_{i+1} - a)\frac{LE_n(h)}{C_n(L,h)}\right) - 1\right), \tag{17}$$

for each $i = 0,1,2,\dots,N-1$.

Proof. The inequality follows from the last inequality in the proof of Theorem 3. Since $(i+1)h = t_{i+1} - t_0 = t_{i+1} - a$, the error bound of this method is obtained from the above inequality, which simplifies to (17).

Remark 3. According to the general theorem of stability for well-posed initial value problems (IVPs), Theorem 3 implies that the general Obreschkoff method described in (15) is stable and consistent.

The primary significance of the error-bound formula presented in Theorem 4 lies in its direct proportionality to the step size h. As a result, reducing the step size should yield proportionally enhanced accuracy in the approximations.

IV. Perturbations of the General Obreschkoff Method

The results of Theorems 3 and 4 do not take into account the impact of round-off errors when selecting the step size. As h decreases, a greater number of computations is required, leading to an increased accumulation of round-off errors. In practice, the difference equation given in (15) is not directly employed to compute the approximation to the solution, denoted as y_i, at a mesh point t_i. Instead, we use an equation of the form

$$\begin{aligned}v_0 &= \alpha + \delta_0,\\ v_{i+1} &= v_i + h\widetilde{B}_1^{(n)}(t_i, v_i) + h\widetilde{B}_2^{(n)}(t_{i+1}, v_{i+1}) + \delta_{i+1},\end{aligned} \tag{18}$$

for each $i = 0,1,2,\dots,N-1$, where

$$\widetilde{B}_1^{(n)}(t_i, v_i) := \sum_{k=1}^{n} \frac{\binom{n}{k}}{\binom{2n}{k}} \frac{h^{k-1}}{k!} f^{(k-1)}(t_i, v_i),$$

and

$$\widetilde{B}_2^{(n)}(t_{i+1}, v_{i+1}) := \sum_{k=1}^{n} (-1)^{k+1} \frac{\binom{n}{k}}{\binom{2n}{k}} \frac{h^{k-1}}{k!} f^{(k-1)}(t_{i+1}, v_{i+1}),$$

for each $i = 0, 1, 2, \ldots, N-1$.

Here, δ_i represents the round-off error associated with the value v_i. By employing techniques similar to those used in the proof of Theorem 3, we can establish an error bound for the finite-precision approximations of y_i as computed by the Obreschkoff method. Consequently, it is possible to formulate an analogous result as follows.

Theorem 5. Let $y(t)$ be the unique solution to the initial-value problem

$$y' = f(t, y), \qquad a \le t \le b, \qquad y(a) = \alpha. \tag{19}$$

Let $v_0, v_1, \ldots, v_N$ be the approximations generated by the Obreschkoff method (18) for some positive integer N. If $|\delta_i| < \delta$ for each $i = 0, 1, \ldots, N$ and the hypotheses of Theorem 3 hold for (19), then

$$\begin{aligned} |y_i - v_i| \le \left(\frac{t(h)}{s(h)} + \frac{\delta C(n,h)}{LhE_n(h)}\right) \cdot \left(e^{\left((t_i-a)\frac{LE_n(h)}{C_n(L,h)}\right)} - 1\right) \\ + |\delta_0| e^{\left((t_i-a)\frac{LE_n(h)}{C_n(L,h)}\right)}. \end{aligned} \tag{20}$$

for each $i = 0, 1, 2, \ldots, N$.

Proof. The proof follows similarly to the proof of Theorem 3, applied to the difference equation (18).

On the other hand, it is important to note that the error bound (20) is no longer linear in h. In fact, since

$$\lim_{h\to 0^+} \left(\frac{t(h)}{s(h)} + \frac{\delta C_n(L,h)}{LhE_n(h)}\right) \to \infty,$$

it follows that as h approaches zero, the error bound increases significantly. Despite this, since $\lim\limits_{h\to 0^+} C_n(L,h)$ is a nonzero finite value, for simplicity, it can be neglected. Thus, as the step size h tends toward infinitesimally small values, the error is expected to escalate. By employing calculus, it is possible to determine a formal lower bound for the step size h. Furthermore, one can precisely determine the minimal value of the round-off error by defining

$$\begin{aligned} R(h) &= \frac{t(h)}{s(h)} + \frac{\delta}{LhE_n(h)} \\ &= \frac{(n!)^2 h^{2n+1}}{(2n)!(2n+1)!hE_n(h)} + \frac{\delta}{hE_n(h)}. \end{aligned} \tag{21}$$

To determine the extreme values of $R(h)$, we compute its derivative

$$R'(h) = \frac{(n!)^2 h^{2n+1} E_n(h) M - 2\delta(2n)!(2n+1)! E_n'(h)}{2(2n)!(2n+1)!(E_n(h))^2}.$$

This implies that

$$\left[(n!)^2 h^{2n+1} E_n(h) M - 2\delta(2n)!(2n+1)!\right] E_n'(h) = 0.$$

Since $E_n'(h) = 0$ is not possible (as $E_n(h)$ is defined above and is not a constant function), we must have

$$(n!)^2 h^{2n+1} E_n(h) M - 2\delta(2n)!(2n+1)! = 0.$$

As $E_n(h)$ is a finite series in h of order n, we introduce the following simplification:

$$h^r = h^{2n+1} \min_{1\le k\le n} \{h^k\},$$

$$\beta_n = \sum_{k=1}^{n} \frac{\binom{n}{k}}{\binom{2n}{k}} \frac{1}{k!},$$

where r is a known fixed positive integer that depends on the order of the Obreschkoff method used and does not exceed $3n+1$. Thus, we obtain

$$\begin{aligned} h^{2n+1} E_n(h) &= \sum_{k=1}^{n} \frac{\binom{n}{k}}{\binom{2n}{k}} \frac{h^{2n+1+k}}{k!} \\ &\ge \frac{2\delta(2n)!(2n+1)!}{(n!)^2 M} \ge h^r \beta_n. \end{aligned}$$

Hence, we conclude that

$$h^r \le \frac{2\delta(2n)!(2n+1)!}{(n!)^2 M \beta_n},$$

or equivalently,

$$h \le \left(\frac{2\delta(2n)!(2n+1)!}{(n!)^2 M \beta_n}\right)^{\frac{1}{r}}.$$

This implies that $R'(h) < 0$, meaning that $R(h)$ is decreasing. Following the same procedure but setting

$$h^r = h^{2n+1} \max_{1\le k\le n} \{h^k\},$$

suggests that

$$h \ge \left(\frac{2\delta(2n)!(2n+1)!}{(n!)^2 M \beta_n}\right)^{\frac{1}{r}},$$

which implies $R'(h) > 0$, meaning that $R(h)$ is increasing. Hence, the minimal value of $R(h)$ occurs when

$$h = \left(\frac{2\delta(2n)!(2n+1)!}{(n!)^2 M \beta_n}\right)^{\frac{1}{r}}, \qquad n = 1, 2, \ldots$$

As the step size h is reduced beyond this critical value, the total error in the approximation tends to increase. Nevertheless, it is important to note that, under typical circumstances, the magnitude of the error, denoted by δ, remains sufficiently small. Consequently, this established lower bound for h does not significantly impact the efficacy or accuracy of the Obreschkoff method in its computational implementation. Despite the theoretical considerations regarding the escalation of error with decreasing h, the practical application of the Obreschkoff method remains robust within the determined range of step sizes.

V. Numerical Experiments

In this section, we apply the Obreschkoff method of order 8 (16) with various step sizes to several initial value problems (IVPs).

Example 1. The Obreschkoff method (16) is employed to approximate the solution of the initial-value problem

$$y'(t) = y - t^2 + 1, \qquad 0 \le t \le 2, \qquad y(0) = 0.5. \qquad (22)$$

The specific parameters are set as follows: $N = 10$, $h = 0.2$, $t_i = 0.2i$, and $w_0 = 1$. This approximation is then compared with the exact solution given by

$$y(t) = (t+1)^2 - 0.5e^t.$$

Furthermore, a comparison is made between the classical Runge–Kutta (RK) approach and our approximation. Specifically, Fig. 1 illustrates the exact solution compared with second-order methods using a step size of $h = 0.2$. Meanwhile, Fig. 2 and Table I present the absolute errors of the Obreschkoff and RK methods of orders 8 and 6, respectively, with the same step size.

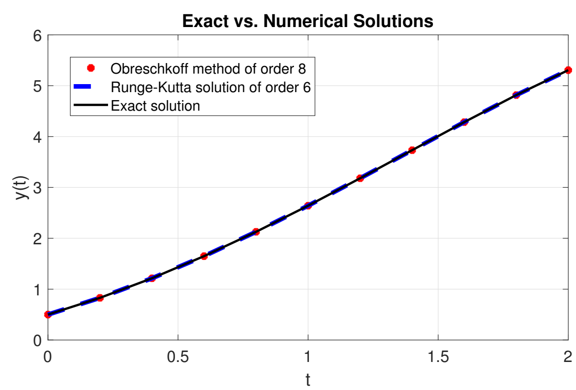

Fig. 1. Comparison of the exact solution with the Obreschkoff and Runge–Kutta methods of orders 8 and 6, respectively, with step size $h = 0.2$, applied in Example 1

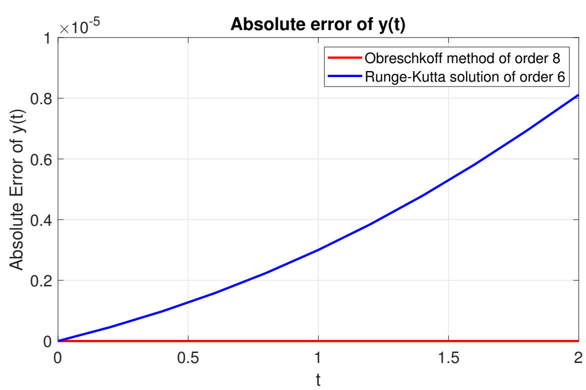

Fig. 2. Absolute errors of the Obreschkoff and Runge–Kutta methods of orders 8 and 6, respectively, with step size $h = 0.2$, applied in Example 1

As observed, the Obreschkoff method (16) provides significantly better approximations compared to the well-known Runge–Kutta (RK) method of order 6. Fig. 1 and Fig. 2 illustrate the comparison between the approximate solutions obtained using both methods, along with their corresponding absolute errors for a step size of $h = 0.2$.

TABLE I. Comparison of absolute errors in the Runge–Kutta (RK) method of order 6 and the Obreschkoff Method (OM) of order 8 applied in Example 3 with step size $h = 0.2$

t_i	**RK Error** $\times 10^{-5}$	**OM Error** $\times 10^{-12}$
0.0	0.0000000	0.0000000
0.2	0.0456003	0.0124344
0.4	0.0977679	0.0304201
0.6	0.1571755	0.0548450
0.8	0.2244720	0.0897060
1.0	0.3002382	0.1376676
1.2	0.3849260	0.2002842
1.4	0.4787718	0.2868816
1.6	0.5816794	0.3996802
1.8	0.6930614	0.5515587
2.0	0.8116256	0.7460698

Example 2. The Obreschkoff method (16) is employed to approximate the solution of the initial-value problem

$$y'(t) = t\exp(y), \qquad 0 \le t \le 0.7, \qquad y(0) = 1. \qquad (23)$$

The specific parameters are set as follows: $N = 10$, $h = 0.07$, $t_i = 0.07i$, and $w_0 = 1$. This approximation is then compared with the exact solution given by

$$y(t) = -\ln\left(\exp(-1) - \frac{1}{2}t^2\right).$$

Furthermore, a comparison is made between the classical Runge–Kutta (RK) approach and our approximation. Specifically, Fig. 3 illustrates the exact solution compared with second-order methods using a step size of $h = 0.07$, while Fig. 4 and Table II present the absolute errors of the Obreschkoff and RK methods of orders 8 and 6, respectively, with the same step size.

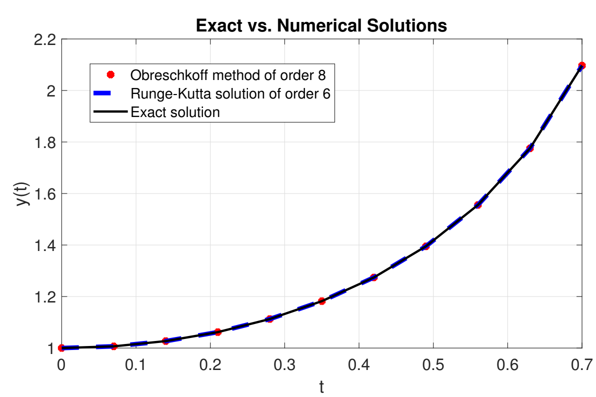

Fig. 3. Comparison of the exact solution with the Obreschkoff and Runge–Kutta methods of orders 8 and 6, respectively, with step size $h = 0.07$, applied in Example 2

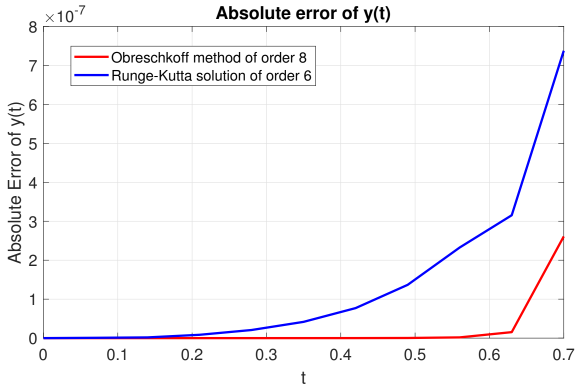

Fig. 4. Absolute errors of the Obreschkoff and Runge–Kutta methods of orders 8 and 6, respectively, with step size $h = 0.07$, applied in Example 2

TABLE II. COMPARISON OF ABSOLUTE ERRORS IN THE RUNGE–KUTTA (RK) METHOD OF ORDER 6 AND THE OBRESCHKOFF METHOD (OM) OF ORDER 8 APPLIED IN EXAMPLE 2 WITH STEP SIZE $h = 0.07$

t_i	**RK Error** $\times 10^{-6}$	**OM Error** $\times 10^{-6}$
0.0	0.00000000	0.00000000
0.07	0.00079991	0.00000019
0.14	0.00185132	0.00000095
0.21	0.00853944	0.00000298
0.28	0.02081735	0.00000859
0.35	0.04176154	0.00002574
0.42	0.07708550	0.00008657
0.49	0.13675361	0.00035061
0.56	0.23260169	0.00187397
0.63	0.31537420	0.01536397
0.70	0.73768121	0.26095318

As observed, the Obreschkoff method (16) provides significantly better approximations compared to the well-known Runge–Kutta (RK) methods.

Example 3. The Obreschkoff method (16) is employed to approximate the solution of the initial-value problem

$$y'(t) = y^2, \qquad 0 \le t \le 0.9, \qquad y(0) = 1. \tag{24}$$

The specific parameters are set as follows: $N = 10$, $h = 0.09$, $t_i = 0.09i$, and $w_0 = 1$. This approximation is then compared with the exact solution given by

$$y(t) = \frac{1}{1-t}.$$

Furthermore, a comparison is made between the Runge–Kutta (RK) method and our approximation. Specifically, Fig. 5 illustrates the exact solution compared with second-order methods using a step size of $h = 0.09$, while Fig. 6 and Table III present the absolute errors of the Obreschkoff and RK methods of orders 8 and 6, respectively, with the same step size.

To enhance our outcomes and improve the Obreschkoff method (16), we consider the following example.

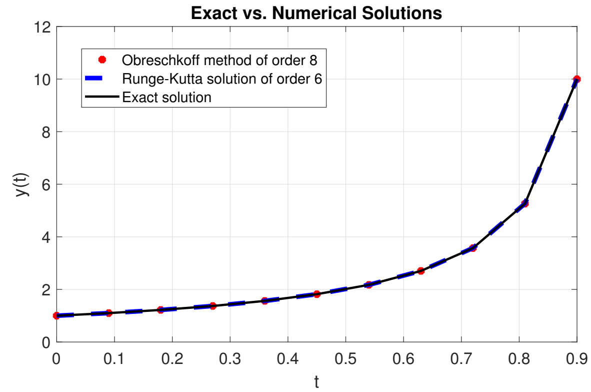

Fig. 5. Comparison of the exact solution with the Obreschkoff and Runge–Kutta methods of orders 8 and 6, respectively, with step size $h = 0.09$, applied in Example 3

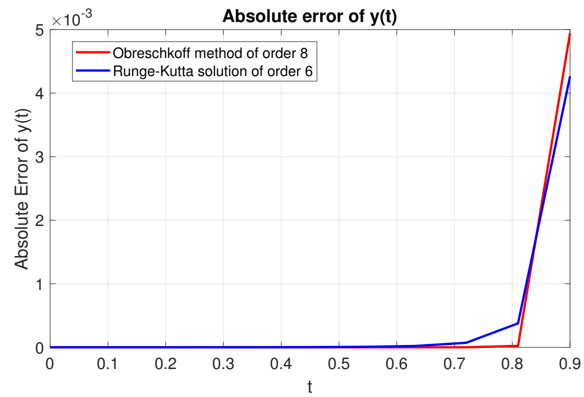

Fig. 6. Absolute errors of the Obreschkoff and Runge–Kutta methods of orders 8 and 6, respectively, with step size $h = 0.09$, applied in Example 3

TABLE III. COMPARISON OF ABSOLUTE ERRORS IN THE RUNGE–KUTTA (RK) METHOD OF ORDER 6 AND THE OBRESCHKOFF METHOD (OM) OF ORDER 8 APPLIED IN EXAMPLE 3 WITH STEP SIZE $h = 0.09$

t_i	**RK Error**	**OM Error**
0.0	0.000000000000	0.000000000000
0.09	0.000000078166	0.000000000009
0.18	0.000000234783	0.000000000038
0.27	0.000000556930	0.000000000130
0.36	0.000001253147	0.000000000454
0.45	0.000002875355	0.000000001817
0.54	0.000007090867	0.000000009054
0.63	0.000019952035	0.000000062989
0.72	0.000070376682	0.000000739863
0.81	0.000376619449	0.000021780352
0.90	0.004264157022	0.004944160607

Example 4. The Obreschkoff method (16) is employed to approximate the solution of the system of linear initial-value problems

$$\begin{cases} z_1'(s) = z_2, & z_1(0) = 1, \\ z_2'(s) = -z_1 - 2e^s + 1, & z_2(0) = 0, \\ z_3'(s) = -z_1 - e^s + 1, & z_3(0) = 1, \end{cases}$$

for $0 \le s \le 2$, with specific parameters set as follows: $N = 10$,

$h = 0.2$, and $t_i = 0.2i$. This approximation is then compared with the exact solution given by

$$\begin{cases} z_1(s) = \cos(s) + \sin(s) - e^s + 1, \\ z_2(s) = -\sin(s) + \cos(s) - e^s, \\ z_3(s) = -\sin(s) + \cos(s). \end{cases}$$

Furthermore, a comparison is made between the classical Runge–Kutta (RK) method and our approximation. Specifically, Fig. 7, Fig. 9, and Fig. 11 illustrate the exact solution compared with the Obreschkoff and RK methods for a step size of $h = 0.2$. Meanwhile, Fig. 8, Fig. 10, and Fig. 12, along with Tables IV, V, and VI, present the absolute errors of the Obreschkoff and RK methods of orders 8 and 6, respectively, using the same step size. Overall, the Obreschkoff method provides remarkably accurate approximations compared to the RK method.

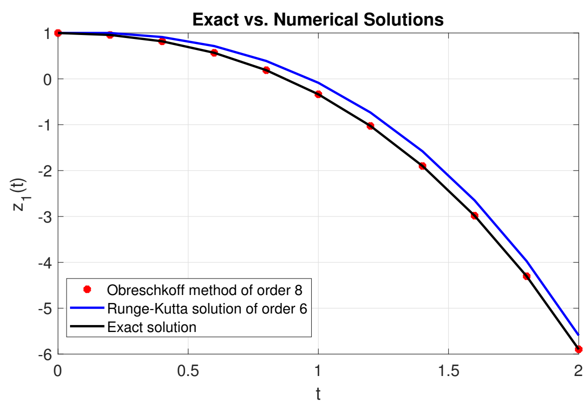

Fig. 7. The exact solution of $z_1(s)$ compared with the Obreschkoff and Runge–Kutta methods of orders 8 and 6, respectively, with step size $h = 0.2$ applied in Example 4

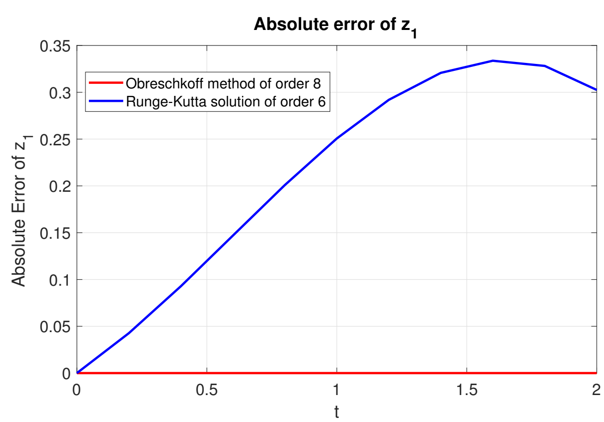

Fig. 8. Absolute errors of the Obreschkoff and Runge–Kutta methods of orders 8 and 6, respectively, with step size $h = 0.2$ applied in Example 4 for $z_1(s)$

VI. RECOMMENDATION

Based on comprehensive analytical and numerical assessments, it is conclusively established that the Obreschkoff method excels in approximating solutions to both linear and nonlinear Initial Value Problems (I.V.P.s) when compared to the Runge–Kutta (RK) method. The demonstrated superiority of the Obreschkoff method extends beyond mere similarity, revealing heightened stability and accelerated convergence. This empirical evidence underscores the method's robustness and efficiency in handling diverse cases in mathematical modeling and analysis.

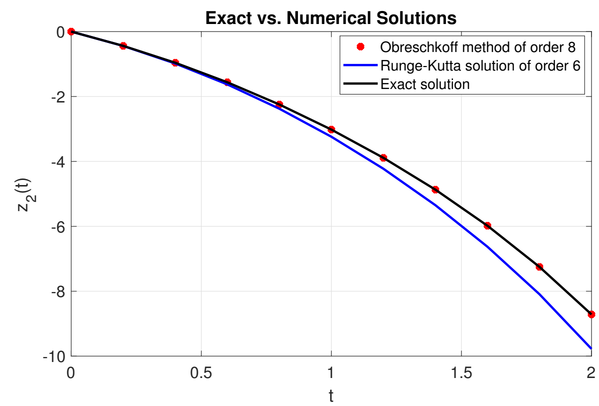

Fig. 9. The exact solution of $z_2(s)$ compared with the Obreschkoff and Runge–Kutta methods of orders 8 and 6, respectively, with step size $h = 0.2$ applied in Example 4

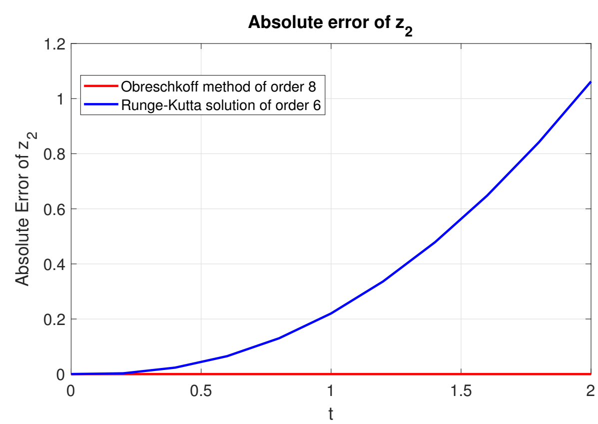

Fig. 10. Absolute errors of the Obreschkoff and Runge–Kutta methods of orders 8 and 6, respectively, with step size $h = 0.2$ applied in Example 4 for $z_2(s)$

Furthermore, the Obreschkoff method presents distinct advantages over the RK method, particularly in cases involving intricate or rapidly changing dynamics, or when an analytical solution is required. Notably, it requires only three derivatives to achieve a solution with an error of order $O(h^8)$. Its enhanced stability ensures a more reliable approximation of solutions, while the accelerated convergence significantly reduces the computational burden. These features position the Obreschkoff method as an invaluable tool for researchers, scientists, and practitioners seeking precise and efficient solutions to a broad spectrum of linear and nonlinear I.V.P.s.

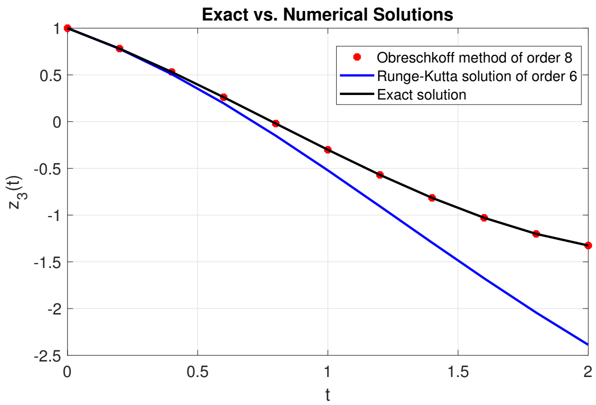

Fig. 11. The exact solution of $z_3(s)$ compared with the Obreschkoff and Runge–Kutta methods of orders 8 and 6, respectively, with step size $h = 0.2$ applied in Example 4

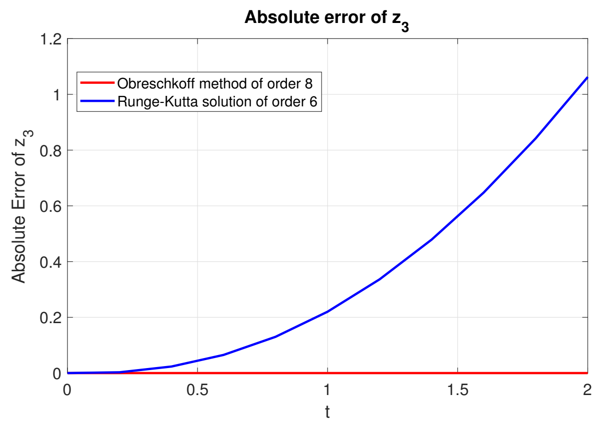

Fig. 12. Absolute errors of the Obreschkoff and Runge–Kutta methods of orders 8 and 6, respectively, with step size $h = 0.2$ applied in Example 4 for $z_3(s)$

On the other hand, it is well known that Taylor's method of order n can be accurate; however, its main drawback lies in the computational complexity of evaluating higher-order derivatives. For this reason, researchers often prefer the Runge–Kutta method, which typically provides the same or even greater accuracy without explicitly computing high-order derivatives. In this work, we have proposed a method that attains an accuracy of order 8 while requiring only three derivatives, unlike the Taylor method, which demands eight derivatives for the same order of accuracy. The most significant advantage of our proposed method is its ability to achieve a high level of precision while minimizing computational cost, making it more efficient and practical for various applications.

The Obreschkoff method has demonstrated remarkable performance compared to other methods, particularly the widely used RK method of order 6, as shown in Example 4. There is no doubt regarding the efficiency of the Obreschkoff method (16); indeed, this method provides outstanding approximations for linear systems of I.V.P.s, especially when an analytical solution is required.

In light of these findings, this work strongly recommends the adoption of the Obreschkoff method (16) as a preferred approach for solving I.V.P.s. Its proven advantages not only contribute to the advancement of numerical techniques but also pave the way for more accurate and efficient solutions across various scientific and engineering disciplines.

TABLE IV. Comparison of absolute errors in the Runge–Kutta (RK) method of order 6 and the Obreschkoff Method (OM) of order 8 applied in Example 4 with step size $h = 0.2$ for $z_1(s)$

t_i	**RK Error**	**OM Error** $(\times 10^{-11})$
0.0	0.0000000	0.0000000
0.2	0.0426668	0.0008770
0.4	0.0927842	0.0036082
0.6	0.1468497	0.0084821
0.8	0.2008820	0.0156014
1.0	0.2506502	0.0250022
1.2	0.2919269	0.0366373
1.4	0.3207555	0.0506261
1.6	0.3337148	0.0667910
1.8	0.3281705	0.0850874
2.0	0.3024980	0.1054267

TABLE V. Comparison of absolute errors in the Runge–Kutta (RK) method of order 6 and the Obreschkoff Method (OM) of order 8 applied in Example 4 with step size $h = 0.2$ for $z_2(s)$

t_i	**RK Error**	**OM Error** $(\times 10^{-12})$
0.0	0.0000000	0.0000000
0.2	0.0027999	0.0436872
0.4	0.0234673	0.0918154
0.6	0.0650996	0.1405542
0.8	0.1301210	0.1856292
1.0	0.2202233	0.2229327
1.2	0.3363741	0.2486899
1.4	0.4788972	0.2611244
1.6	0.6476335	0.2557953
1.8	0.8421832	0.2318145
2.0	1.0622309	0.1882938

TABLE VI. Comparison of absolute errors in the Runge–Kutta (RK) method of order 6 and the Obreschkoff Method (OM) of order 8 applied in Example 4 with step size $h = 0.2$ for $z_3(s)$

t_i	**RK Error**	**OM Error** $(\times 10^{-12})$
0.0	0.0000000	0.0000000
0.2	0.0027999	0.0213162
0.4	0.0234673	0.0422994
0.6	0.0650996	0.0578981
0.8	0.1301210	0.0622939
1.0	0.2202234	0.0500710
1.2	0.3363742	0.0156541
1.4	0.4788973	0.0467403
1.6	0.6476337	0.1423305
1.8	0.8421834	0.2766675
2.0	1.0622312	0.4545253

VII. Conclusion and Future Work

This paper presented the Obreschkoff method as a high-order numerical approach for solving initial value problems (IVPs).

Theoretical analysis confirmed its stability and convergence, while numerical comparisons demonstrated superior accuracy over classical Runge-Kutta methods. Despite its efficiency, future work will focus on adaptive step-size implementation, extending the method to stiff and large-scale systems, optimizing computational complexity, and exploring GPU parallelization. Additionally, further comparisons with modern numerical techniques and applications in real-world problems will provide deeper insights into its practical utility.

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