A Comparative Analysis of Numerical Techniques: Euler-Maclaurin vs. Runge-Kutta Methods

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***Abstract*—This study introduces a novel higher-order implicit correction method derived from the Euler-Maclaurin formula to enhance the approximation of initial value problems. The proposed method surpasses the Runge-Kutta approach in accuracy, stability, and convergence. An error bound is established to demonstrate its theoretical reliability. To validate its effectiveness, numerical experiments are conducted, showcasing its superior performance compared to conventional methods. The results consistently confirm that the proposed method outperforms the Runge-Kutta method across various practical applications.**

***Keywords*—Euler-Maclaurin Formula, Runge-Kutta Method, Ode, Darboux's Formula, Approximations**

I. Introduction

In our present era, characterized by rapid advancements in both experimental and applied sciences, the scope of scientific exploration continues to expand. A significant aspect of this progress is the swift evolution of artificial intelligence, a transformative force with the potential to tackle complex mathematical challenges. In the dynamic field of differential equations, researchers are actively working to refine and modernize classical methods for approximating both initial and boundary value problems [1]–[3]. To see more studies about the IVPs, BVPs and their generalizations, applications, and more, the reader may refer to the references [4]–[24].

While the Runge-Kutta method remains the predominant technique for solving differential equations, researchers find themselves balancing tradition and innovation. Widely regarded as a benchmark, this method serves as a standard against which new approaches are evaluated, particularly in the complex domain of chaotic systems. However, in an era of rapid scientific advancements, it is essential not only to acknowledge established methodologies but also to explore and extend beyond their limitations.

This contemporary era necessitates the proactive development of novel approaches for approximating Ordinary Differential Equations (O.D.E.) with greater precision and efficiency. The demand for advancements in computational techniques is increasingly evident as we strive to gain deeper insights into complex mathematical models and systems (see [25]–[31] for future directions in this field). In this pursuit, researchers are encouraged to explore uncharted methodologies that not only surpass the reliability of the Runge-Kutta method but also align with the evolving needs of modern scientific inquiry. Standing at the intersection of tradition and innovation, our goal extends beyond mere comparison—toward pioneering new frameworks that redefine the landscape of mathematical approximation in the age of artificial intelligence.

After reviewing established methodologies, our exploration now culminates in the introduction of a groundbreaking approach for approximating solutions to Initial Value Problems (I.V.P.). This innovative method aims to achieve a refined balance between precision and computational efficiency, presenting a compelling alternative to conventional techniques. As we embark on this transformative journey, we invite readers to engage in unraveling the complexities of numerical methods, fostering a new era in I.V.P. approximations. For further insights, we recommend referring to [32]–[36].

The Euler-Maclaurin formula, a fundamental result in mathematical analysis, was independently developed by Euler [37] and Maclaurin [38] in the 18th century. Euler's motivation was to establish a connection between discrete sums and continuous integrals, while Maclaurin extended and refined these ideas. Their combined contributions resulted in a powerful formula that remains a cornerstone in numerical analysis and asymptotic approximations. Specifically, if the function $f(x)$ is analytic over the integration region, the Euler-Maclaurin formula is given by

$$\begin{aligned}\sum_{k=1}^{n-1} f(k) &= \int_0^n f(x)\,dx - \tfrac{f(0)+f(n)}{2}\\ &+ \sum_{k=1}^{\infty} \tfrac{B_{2k}}{(2k)!}\left[f^{(2k-1)}(n) - f^{(2k-1)}(0)\right].\end{aligned}$$

An elementary perspective on the Euler-Maclaurin formula is extensively discussed in [39]. The elegance of this formula lies in its derivation, which is fundamentally based on integration by parts. By strategically applying this technique, Euler and Maclaurin established a powerful connection between discrete sums and continuous integrals. The derivation involves transforming summations into integral expressions, carefully handling boundary terms, and incorporating correction terms, ultimately leading to a remarkably expressive formulation. This process exemplifies their ingenuity in bridging discrete and continuous mathematics. Over time, the Euler-Maclaurin formula has garnered significant attention, inspiring extensive research and leading to various alternative formulations and extensions.

Darboux provided an alternative derivation of the Euler-Maclaurin formula by applying the mean value theorem to the integral terms. This approach offers a fresh perspective, reinforcing the connection between discrete and continuous processes through the lens of classical analysis. Darboux's insight deepens our understanding of the formula, highlighting the diverse mathematical pathways that lead to its elegant expression. Throughout this work, we consider a real interval I, where $a, b \in I^{\circ}$ (the interior of I) with $a < b$. Additionally, let $\mathcal{P}_n(I)$ denote the class of polynomials of degree at most n defined on $I \subseteq \mathbb{R}$.

The origins of the Euler-Maclaurin formula can be traced back to the celebrated Darboux formula. Let $f(x)$ be analytic over the interval $[a, x]$, and let $\phi(t) \in \mathcal{P}_n$. For $t \in [0, 1]$, differentiation yields

$$\begin{aligned}\tfrac{d}{dt}\sum_{k=1}^{n} &(-1)^k (x-a)^k \phi^{(n-k)}(t) f^{(k)}(a + t(x-a))\\ &= -(x-a)\,\phi^{(n)}(t) f'(a + t(x-a))\\ &+ (-1)^n (x-a)^{n+1} \phi(t) f^{(n+1)}(a + t(x-a)).\end{aligned} \tag{1}$$

Since $\phi^{(n)}(t) = \phi^{(n)}(0) = \text{constant}$, integrating from 0 to 1 with respect to t yields

$$\begin{aligned}\phi^{(n)}(0)\left[f(x) - f(a)\right] &= \sum_{k=1}^{n} (-1)^{k-1} (x-a)^k\\ &\times \left\{\phi^{(n-k)}(1) f^{(k)}(x) - \phi^{(n-k)}(0) f^{(k)}(a)\right\}\\ &+ (-1)^n (x-a)^{n+1} \int_0^1 \phi(t) f^{(n+1)}(a + t(x-a))\,dt,\end{aligned} \tag{2}$$

Which is known as Darboux's formula (see [40]). A detailed discussion of this formula is also provided in [41].

The Euler-Maclaurin formula serves as a cornerstone in mathematical analysis, offering a bridge between discrete sums and continuous integrals. Its significance extends beyond its historical origins, influencing contemporary mathematics, physics, and engineering. The formula's ability to simplify complex computations and yield precise approximations makes it an indispensable tool across various scientific disciplines. Its continued relevance underscores its profound impact, providing essential insights into both discrete and continuous mathematical structures.

In his formulation, Darboux arrived at an expansion that is no less significant than the celebrated Euler-Maclaurin formula itself. Indeed, the following expansion holds [40]:

$$\begin{aligned}(x-a)f'(a) &= f(x) - f(a) - \tfrac{x-a}{2}\left[f'(x) - f'(a)\right]\\ &+ \sum_{m=1}^{n-1} \tfrac{(-1)^{m-1} B_m (x-a)^{2m}}{(2m)!}\left[f^{(2m)}(x) - f^{(2m)}(a)\right]\\ &- R_n(f, B_{2n}),\end{aligned}$$

where the remainder term is given by

$$\begin{aligned}&R_n(f, B_{2n})\\ &= \tfrac{(x-a)^{2n+1}}{(2n)!}\int_0^1 B_{2n}(t) f^{(2n+1)}(a + t(x-a))\,dt,\end{aligned} \tag{3}$$

where $B_k(t)$ $(k = 1, 2, 3, \dots)$ are the Bernoulli polynomials, and B_k are the Bernoulli numbers. Since all odd Bernoulli numbers B_{2k-1} $(k = 1, 2, \dots)$ are zero, the above expansion simplifies to

$$\begin{aligned}f(x) &= f(a) + (x-a) f'(a) + \tfrac{(x-a)}{2}\left[f'(x) - f'(a)\right]\\ &- \sum_{m=1}^{n-1} (-1)^{m-1} \tfrac{B_{2m}(x-a)^{2m}}{(2m)!}\left[f^{(2m)}(x) - f^{(2m)}(a)\right]\\ &+ R_n(f, B_{2n}).\end{aligned} \tag{4}$$

Based on this formulation, this work derives a general higher-order implicit method that surpasses the Runge-Kutta methods in terms of accuracy. An error bound for the Euler-Maclaurin higher-order method is established, demonstrating its stability, convergence, and superior efficiency compared to conventional Runge-Kutta methods. To substantiate these claims, numerical experiments are conducted, highlighting the exceptional performance of the proposed method over traditional well-established techniques.

II. The Euler-Maclaurin Method for Approximating Solutions of I.V.P.

This section presents a novel approach for approximating the solution of the well-posed initial-value problem

$$\frac{dy}{dt} = f(t, y), \qquad a \le t \le b, \qquad y(a) = \alpha. \tag{5}$$

Assume that the solution $y(t)$ possesses $(2n+1)$-continuous derivatives. Expanding $y(t)$ using its $(2n)$-th order Euler-Maclaurin expansion about t_i and evaluating at t_{i+1}, we obtain

$$\begin{array}{c} y\left(t_{i+1}\right)=y\left(t_i\right)+\left(t_{i+1}-t_i\right)y'\left(t_i\right) \\ +\frac{\left(t_{i+1}-t_i\right)}{2}\left[y'\left(t_{i+1}\right)-y'\left(t_i\right)\right]-\sum\limits_{m=1}^{n-1}(-1)^{m-1} \\ \times\frac{B_{2m}\left(t_{i+1}-t_i\right)^{2m}}{(2m)!}\left[y^{(2m)}\left(t_{i+1}\right)-y^{(2m)}\left(t_i\right)\right] \\ +\frac{\left(t_{i+1}-t_i\right)^{2n+1}}{(2n)!}\int_0^1 B_{2n}(s)\,y^{(2n+1)}\left(t_i+s\left(t_{i+1}-t_i\right)\right)ds \end{array} \tag{6}$$

We begin by establishing the assumption that the distribution of mesh points is uniform across the interval $[a,b]$. This requirement is ensured by selecting a positive integer N, from which the mesh points are defined as

$$t_i=a+ih, \qquad \text{for} \quad i=0,1,2,\ldots,N.$$

Here, h represents the step size or the uniform spacing between consecutive points, given by

$$h=\frac{b-a}{N}=t_{i+1}-t_i.$$

Assuming that the unique solution to (5) possesses $(2n+1)$ continuous derivatives on $[a,b]$, this holds for each $i=0,1,2,\ldots,N-1$. Furthermore, since $y(t)$ satisfies the differential equation (6), successive differentiation of $y(t)$ yields

$$y'(t)=f(t,y(t)),\ldots,y^{(k)}(t)=f^{(k-1)}(t,y(t)).$$

Substituting these expressions into (6), we obtain

$$\begin{array}{c} y\left(t_{i+1}\right)=y\left(t_i\right)+hf\left(t_i,y\left(t_i\right)\right) \\ +\frac{h}{2}\left[f\left(t_{i+1},y\left(t_{i+1}\right)\right)-f\left(t_i,y\left(t_i\right)\right)\right] \\ -\sum\limits_{m=1}^{n-1}(-1)^{m-1}\frac{B_{2m}h^{2m}}{(2m)!} \\ \times\left[f^{(2m-1)}\left(t_{i+1},y\left(t_{i+1}\right)\right)-f^{(2m-1)}\left(t_i,y\left(t_i\right)\right)\right]. \end{array} \tag{7}$$

The difference-equation method corresponding to (7) is derived by omitting the remainder term involving ξ_i, yielding

$$\begin{array}{c} w_0=\alpha, \\ w_{i+1}=w_i+hf\left(t_i,y\left(t_i\right)\right) \\ +\frac{h}{2}\left[f\left(t_{i+1},y\left(t_{i+1}\right)\right)-f\left(t_i,y\left(t_i\right)\right)\right] \\ -h\mathcal{M}^{(n-1)}\left(w_i,w_{i+1}\right), \end{array} \tag{8}$$

for $i=0,1,2,\ldots,N-1$, where

$$\begin{array}{c} \mathcal{M}^{(n-1)}\left(w_i,w_{i+1}\right):=\sum\limits_{m=1}^{n-1}(-1)^{m-1}\frac{B_{2m}h^{2m-1}}{(2m)!} \\ \times\left[f^{(2m-1)}\left(t_{i+1},y\left(t_{i+1}\right)\right)-f^{(2m-1)}\left(t_i,y\left(t_i\right)\right)\right]. \end{array}$$

In particular, we focus on the specific case of (8) presented in the following section.

A. Euler-Maclaurin Method of Order 11

By setting $n=5$ in (8), we obtain the following recurrence relation:

$$\begin{array}{c} w_0=\alpha, \\ w_{i+1}=w_i+hf\left(t_i,w_i\right)+\frac{h}{2}\left[f\left(t_{i+1},w_{i+1}\right)-f\left(t_i,w_i\right)\right] \\ +\frac{h^2}{12}\left[f'\left(t_{i+1},w_{i+1}\right)-f'\left(t_i,w_i\right)\right] \\ -\frac{h^4}{720}\left[f'''\left(t_{i+1},w_{i+1}\right)-f'''\left(t_i,w_i\right)\right] \\ +\frac{h^6}{30240}\left[f^{(5)}\left(t_{i+1},w_{i+1}\right)+f^{(5)}\left(t_i,w_i\right)\right] \\ -\frac{h^8}{1209600}\left[f^{(7)}\left(t_{i+1},w_{i+1}\right)-f^{(7)}\left(t_i,w_i\right)\right], \end{array} \tag{9}$$

for each $i=0,1,2,\ldots,N-1$.

Proposition 1. The Euler-Maclaurin Method given in (9) is of order 11.

Proof. Substituting the exact solution into the Taylor series expansion and simplifying, we obtain

$$\begin{array}{c} y\left(t_{i+1}\right)-y\left(t_i\right)-hf\left(t_i,y\left(t_i\right)\right) \\ -\frac{h}{2}\left[f\left(t_{i+1},y\left(t_{i+1}\right)\right)-f\left(t_i,y\left(t_i\right)\right)\right] \\ +\frac{h^2}{12}\left[f'\left(t_{i+1},y\left(t_{i+1}\right)\right)-f'\left(t_i,y\left(t_i\right)\right)\right] \\ -\frac{h^4}{720}\left[f'''\left(t_{i+1},y\left(t_{i+1}\right)\right)-f'''\left(t_i,y\left(t_i\right)\right)\right] \\ +\frac{h^6}{30240}\left[f^{(5)}\left(t_{i+1},y\left(t_{i+1}\right)\right)-f^{(5)}\left(t_i,y\left(t_i\right)\right)\right] \\ -\frac{h^8}{1209600}\left[f^{(7)}\left(t_{i+1},y\left(t_{i+1}\right)\right)-f^{(7)}\left(t_i,y\left(t_i\right)\right)\right] \\ =O\left(h^{11}\right). \end{array}$$

Thus, the local truncation error is $O(h^{11})$, confirming that (9) is of order 11.

Remark 1. In general, by employing mathematical induction, one can observe that the Euler-Maclaurin method achieves an order of accuracy $O\left(h^{2n+1}\right)$.

III. CONVERGENCE AND STABILITY OF THE GENERAL EULER-MACLAURIN METHOD

To establish the convergence and derive the error bound for the general Euler-Maclaurin method (8), we first introduce the following key lemma from [42].

Lemma 1. Let s and t be positive real numbers, and let $\{a_i\}_{i=1}^{k}$ be a sequence satisfying $a_0\geq -t/s$ and

$$a_{i+1}\leq\exp\left((1+i)s\right)\left(a_0+\frac{t}{s}\right)-\frac{t}{s}.$$

The following result establishes the convergence of the Euler-Maclaurin method of order $2n$ and provides a corresponding error bound.

Theorem 1. Suppose that the derivatives $f^{(k)}$ for $0\leq k\leq 2n-1$ are continuous and satisfy the Lipschitz condition with constant L_k on the domain

$$D:=\{(t,y)\mid a\leq t\leq b,-\infty<y<\infty\}.$$

Additionally, assume that there exists a constant M such that

$$\left|f^{(2n)}\left(t,y(t)\right)\right|\leq M,\quad \text{for all } t\in\left[a,b\right],$$

where $y(t)$ is the unique solution to the initial-value problem

$$y'=f\left(t,y\right),\quad a\leq t\leq b,\quad y\left(a\right)=\alpha.$$

Let $w_0,w_1,\ldots,w_N$ be the approximations generated by the Euler-Maclaurin method (8) for some positive integer N. Then, the general Euler-Maclaurin method described in (8) is convergent.

Proof. For $i=0$, the assertion holds trivially since $y(t_0)=w_0=\alpha$. For $i\geq 1$, using the Euler-Maclaurin expansion from (6), we obtain

$$\begin{gathered}y\left(t_{i+1}\right)=y\left(t_i\right)\\ +hf\left(t_i,y\left(t_i\right)\right)+\tfrac{h}{2}\left[f\left(t_{i+1},y\left(t_{i+1}\right)\right)-f\left(t_i,y\left(t_i\right)\right)\right]\\ -\sum_{m=1}^{n-1}\left(-1\right)^{m-1}\tfrac{B_{2m}h^{2m}}{(2m)!}\\ \times\left[f^{(2m-1)}\left(t_{i+1},y\left(t_{i+1}\right)\right)-f^{(2m-1)}\left(t_i,y\left(t_i\right)\right)\right]\\ +\tfrac{h^{2n+1}}{(2n)!}\int_0^1 B_{2n}\left(s\right)f^{(2n)}\left(t_i+s\left(t_{i+1}-t_i\right)\right)ds,\end{gathered}$$

for $i=0,1,\ldots,N-1$.
Similarly, from (8), the numerical approximation satisfies

$$\begin{gathered}w_{i+1}=w_i+hf\left(t_i,w_i\right)+\tfrac{h}{2}\left[f\left(t_{i+1},w_{i+1}\right)-f\left(t_i,w_i\right)\right]\\ +-\sum_{m=1}^{n-1}\left(-1\right)^{m-1}\tfrac{B_{2m}h^{2m}}{(2m)!}\\ \times\left[f^{(2m-1)}\left(t_{i+1},w_{i+1}\right)-f^{(2m-1)}\left(t_i,w_i\right)\right],\end{gathered}$$

for each $i=0,1,2,\ldots,N-1$. Utilizing the notations $y_i=y(t_i)$ and $y_{i+1}=y(t_{i+1})$, we obtain the following by subtracting the numerical scheme from the exact solution:

$$\begin{gathered}y_{i+1}-w_{i+1}=y_i-w_i+h\left[f\left(t_i,y_i\right)-f\left(t_i,w_i\right)\right]\\ +\tfrac{h}{2}\left[f\left(t_{i+1},y_{i+1}\right)-f\left(t_{i+1},w_{i+1}\right)\right]\\ -\tfrac{h}{2}\left[f\left(t_i,y_i\right)-f\left(t_i,w_i\right)\right]\\ -\sum_{m=1}^{n-1}\left(-1\right)^{m-1}\tfrac{B_{2m}h^{2m}}{(2m)!}\\ \times\left[f^{(2m-1)}\left(t_{i+1},y_{i+1}\right)-f^{(2m-1)}\left(t_{i+1},w_{i+1}\right)\right]\\ -\sum_{m=1}^{n-1}\left(-1\right)^{m-1}\tfrac{B_{2m}h^{2m}}{(2m)!}\\ \times\left[f^{(2m-1)}\left(t_i,y_i\right)-f^{(2m-1)}\left(t_i,w_i\right)\right]\\ +\tfrac{h^{2n+1}}{(2n)!}\int_0^1 B_{2n}\left(s\right)f^{(2n)}\left(t_i+s\left(t_{i+1}-t_i\right)\right)ds.\end{gathered}$$

Employing the triangle inequality, we obtain

$$\begin{gathered}\left|y_{i+1}-w_{i+1}\right|\leq\left|y_i-w_i\right|+h\left|f\left(t_i,y_i\right)-f\left(t_i,w_i\right)\right|\\ +\tfrac{h}{2}\left|f\left(t_{i+1},y_{i+1}\right)-f\left(t_{i+1},w_{i+1}\right)\right|\\ +\tfrac{h}{2}\left|f\left(t_i,y_i\right)-f\left(t_i,w_i\right)\right|\\ +\sum_{m=1}^{n-1}\tfrac{B_{2m}h^{2m}}{(2m)!}\\ \times\left|f^{(2m-1)}\left(t_{i+1},y_{i+1}\right)-f^{(2m-1)}\left(t_{i+1},w_{i+1}\right)\right|\\ +\sum_{m=1}^{n-1}\tfrac{B_{2m}h^{2m}}{(2m)!}\\ \times\left|f^{(2m-1)}\left(t_i,y_i\right)-f^{(2m-1)}\left(t_i,w_i\right)\right|\\ +\tfrac{h^{2n}}{(2n)!}\left|f^{(2n)}\left(\mu_i,y\left(\mu_i\right)\right)\right|\int_0^1\left|B_{2n}\left(s\right)\right|ds.\end{gathered}$$

Now, the function $f^{(m-1)}$ $(m=1,2,\ldots,2n-1)$ satisfies the Lipschitz condition in the second variable with a constant denoted as

$$L:=\max_{1\leq m\leq 2n-1}\left\{L_k\right\},$$

and it holds that $\left|f^{(2n+1)}\left(t,y\left(t\right)\right)\right|\leq M$. Thus, we obtain

$$\begin{gathered}\left|y_{i+1}-w_{i+1}\right|\leq\left|y_i-w_i\right|+hL\left|y_i-w_i\right|\\ +\tfrac{h}{2}L\left|y_{i+1}-w_{i+1}\right|+\tfrac{h}{2}L\left|y_i-w_i\right|\\ +L\sum_{m=1}^{n-1}\tfrac{|B_{2m}|h^{2m}}{(2m)!}\left|y_{i+1}-w_{i+1}\right|\\ +L\sum_{m=1}^{n-1}\tfrac{|B_{2m}|h^{2m}}{(2m)!}\left|y_i-w_i\right|\\ +\tfrac{h^{2n}}{(2n)!}M\int_0^1\left|B_{2n}\left(s\right)\right|ds.\end{gathered}$$

Combining the terms, we obtain

$$\begin{gathered}\left|y_{i+1}-w_{i+1}\right|\leq\left(\tfrac{1}{2}hL+L\sum_{m=1}^{n-1}\tfrac{|B_{2m}|h^{2m}}{(2m)!}\right)\\ \times\left|y_{i+1}-w_{i+1}\right|+\left(1+\tfrac{3}{2}hL+L\sum_{m=1}^{n-1}\tfrac{|B_{2m}|h^{2m}}{(2m)!}\right)\\ \times\left|y_i-w_i\right|+\tfrac{h^{2n}}{(2n)!}M|B_{2n}|.\end{gathered}$$

where we used the fact that $\left|B_{2n}\left(s\right)\right|<|B_{2n}|$, see [43].
For simplicity, we define

$$\begin{gathered}S_n\left(L,h\right):=\left(1+\tfrac{3}{2}hL+L\sum_{m=1}^{n-1}\tfrac{|B_{2m}|h^{2m}}{(2m)!}\right),\\ C_n\left(L,h\right):=\left(1-\tfrac{1}{2}hL-L\sum_{m=1}^{n-1}\tfrac{|B_{2m}|h^{2m}}{(2m)!}\right),\end{gathered}$$

and

$$E_n\left(h\right):=2\sum_{m=1}^{n-1}\frac{|B_{2m}|h^{2m-1}}{(2m)!}.$$

Before proceeding further, we note that

$$\begin{aligned}\tfrac{1}{2}LhE_n\left(h\right)&=L\sum_{m=1}^{n-1}\tfrac{|B_{2m}|h^{2m}}{(2m)!}\\ &\leq L\cdot\max_{1\leq m\leq n-1}\left\{h^{2m}\right\}\cdot\sum_{m=1}^{n-1}\tfrac{|B_{2m}|}{(2m)!}\\ &\approx L\cdot\max_{1\leq m\leq n-1}\left\{h^{2m}\right\}\cdot\sum_{m=1}^{n-1}\tfrac{2(2m)!}{(2\pi)^{2m}}\cdot\tfrac{1}{(2m)!}\\ &=K\cdot\left[\tfrac{2}{4\pi^2-1}+\tfrac{8\pi^2}{1-4\pi^2}\cdot\left(\tfrac{1}{4\pi^2}\right)^n\right],\end{aligned}$$

where the last summation was evaluated using Maple Software.

Before this, we utilized the asymptotic approximation of even Bernoulli numbers [43], given by

$$(-1)^{m+1} B_{2m} \approx \frac{2\,(2m)!}{(2\pi)^{2m}}, \quad \text{for every positive integer } m.$$

Moreover, as $n \to \infty$, we obtain

$$\frac{1}{2} LhE_n\,(h) \leq K \cdot \frac{2}{4\pi^2 - 1}.$$

Considering our ultimate interest in allowing $h \to 0^+$, it is reasonable to assume that

$$\frac{1}{2} LhE_n\,(h) < K \cdot \frac{2}{4\pi^2 - 1},$$

where K is some fixed nonzero positive real number, without any adverse consequences. Consequently, we can infer that

$$\begin{aligned} |y_{i+1} - w_{i+1}| &\leq \tfrac{S_n(L,h)}{C_n(L,h)} \cdot |y_i - w_i| + \tfrac{h^{2n} M |B_{2n}|}{(2n)! C_n(L,h)} \\ &= \left(1 + \tfrac{S_n(L,h) - C_n(L,h)}{C_n(L,h)}\right) \cdot |y_i - w_i| \\ &\quad + \tfrac{h^{2n}}{(2n)! C_n(L,h)} M\,|B_{2n}| \\ &= \left(1 + \tfrac{L \cdot h \cdot E_n(h)}{C_n(L,h)}\right) \cdot |y_i - w_i| \\ &\quad + \tfrac{h^{2n}}{(2n)! C_n(L,h)} M\,|B_{2n}|\,. \end{aligned}$$

Employing Lemma 1, with

$$s(h) = \tfrac{L \cdot h \cdot E_n(h)}{C_n(L,h)}, \qquad t(h) = \tfrac{h^{2n}}{(2n)! C_n(L,h)} M\,|B_{2n}|\,,$$

and defining $a_j = |y_j - w_j|$ for each $j = 0, 1, 2, \ldots, N$, we observe that

$$\begin{aligned} |y_{i+1} - w_{i+1}| &\leq \exp\left((i+1) \cdot \tfrac{L \cdot h \cdot E_n(h)}{C_n(L,h)}\right) \\ &\quad \times \left(|y_0 - w_0| + \tfrac{t(h)}{s(h)}\right) - \tfrac{t(h)}{s(h)}. \end{aligned}$$

Since $|y_0 - w_0| = 0$, it follows that

$$\lim_{h \to 0^+} \frac{L \cdot h \cdot E_n(h)}{C_n(L,h)} = 0, \qquad \text{and} \qquad \lim_{h \to 0^+} \frac{t(h)}{s(h)} = 0.$$

Thus, we conclude that

$$\lim_{h \to 0^+} \max_{1 \leq i \leq N} |y_{i+1} - w_{i+1}| = 0,$$

Which implies that w_{i+1} converges to y_{i+1}. Consequently, the Euler-Maclaurin Method of Order $2n$ is convergent as required.

Theorem 2. Under the assumptions of Theorem 1, the error bound for the Euler-Maclaurin method is given by

$$|y_{i+1} - w_{i+1}| \leq \frac{t\,(h)}{s\,(h)} \left(\exp\left((t_{i+1} - a)\,\frac{LhE_n\,(h)}{C_n\,(L,h)}\right) - 1\right), \tag{10}$$

for all $i = 0, 1, 2, \ldots, N-1$.

Proof. This bound follows directly from the final inequality in the proof of Theorem 1. Given that $(i+1)\,h = t_{i+1} - t_0 = t_{i+1} - a$, substituting this into the derived expression yields the result in (10), completing the proof.

Remark 2. Based on the fundamental theorem of stability for well-posed initial value problems (I.V.P.), Theorem (1) establishes that the general Euler-Maclaurin method, as formulated in (9), is both stable and consistent.

A key takeaway from the error bound derived in Theorem 1 is its direct dependence on the step size, h. Consequently, decreasing h leads to a proportional improvement in the accuracy of the computed approximations.

IV. Perturbations in the General Euler-Maclaurin Method

Theorems 1 and 2 establish the convergence and error bound of the Euler-Maclaurin method; however, they do not account for the influence of round-off errors when selecting the step size. As h decreases, the number of required computations increases, which in turn amplifies the accumulation of round-off errors. In practical implementations, the recurrence relation in (8) is not directly used to compute the numerical approximation of the solution y_i at the mesh points t_i. Instead, the following perturbed equation is employed:

$$\begin{aligned} v_0 &= \alpha + \delta_0, \\ v_{i+1} &= v_i + h\widetilde{B}^{(n)}\,(t_i, v_i) + \delta_{i+1}, \end{aligned} \tag{11}$$

for each $i = 0, 1, 2, \ldots, N-1$, where

$$\begin{aligned} &\widetilde{B}^{(n)}\,(t_i, v_i) := f\,(t_i, v_i) + \tfrac{1}{2}\left[f\,(t_{i+1}, v_{i+1}) - f\,(t_i, v_i)\right] \\ &- \sum_{m=1}^{n-1} \tfrac{B_{2m} h^{2m-1}}{(2m)!} \left[f^{(2m-1)}\,(t_{i+1}, v_{i+1}) - f^{(2m-1)}\,(t_i, v_i)\right]. \end{aligned}$$

Here, δ_i represents the round-off error associated with the computed value v_i. By employing techniques similar to those used in the proof of Theorem 1, we can establish an upper bound on the numerical error introduced by finite-precision computations in the Euler-Maclaurin method. Consequently, it is possible to derive an analogous result to the one presented in the following theorem.

Theorem 3. Let $y(t)$ be the unique solution to the initial-value problem

$$y' = f\,(t, y)\,, \qquad a \leq t \leq b, \qquad y\,(a) = \alpha. \tag{12}$$

Let $\{v_i\}_{i=0}^{N}$ be the numerical approximations obtained using the Euler-Maclaurin method (8) for a given positive integer N. Suppose the round-off errors satisfy $|\delta_i| < \delta$ for each $i = 0, 1, \ldots, N$, and the assumptions of Theorem 1 hold for (12). Then, the error bound for the approximations is given by

$$\begin{aligned} |y_i - v_i| &\leq \left(\tfrac{t(h)}{s(h)} + \tfrac{\delta C(n,h)}{LhE_n(h)}\right)\left(e^{\left((t_i - a)\frac{LhE_n(h)}{C_n(L,h)}\right)} - 1\right) \\ &\quad + |\delta_0| e^{\left((t_i - a)\frac{LhE_n(h)}{C_n(L,h)}\right)}, \end{aligned} \tag{13}$$

for all $i = 0, 1, 2, \ldots, N$.

Proof. The proof follows a similar argument as in Theorem 1, but applied to the difference equation (11).

It is important to observe that the error bound given in (13) is no longer linear in h. In fact, we note that

$$\lim_{h\to 0^+}\left(\frac{t\left(h\right)}{s\left(h\right)}+\frac{\delta C_n\left(L,h\right)}{LhE_n\left(h\right)}\right)\to\infty.$$

As the step size h approaches increasingly small values, the error is expected to grow. Furthermore, if h is reduced beyond a certain critical threshold, the total error in the numerical approximation may increase. However, it is essential to recognize that in most practical scenarios, the round-off error δ remains sufficiently small. As a result, this theoretical lower bound on h does not significantly hinder the computational effectiveness or accuracy of the Euler-Maclaurin method. Despite the theoretical implications regarding error accumulation for extremely small values of h, the method remains computationally stable and reliable within a practical range of step sizes.

V. Numerical Experiments

In this section, we apply the Euler-Maclaurin method of order 11 to various initial-value problems (I.V.P.s) using different step sizes.

Example 1. To illustrate the performance of the Euler-Maclaurin method of order 11 (9), we approximate the solution of the initial-value problem

$$y'\left(t\right)=y-t^2+1,\qquad 0\le t\le 2,\qquad y(0)=0.5. \tag{14}$$

For this experiment, we set the parameters as follows: $N = 10$, step size $h = 0.2$, discrete points $t_i = 0.2i$, and initial condition $w_0 = 0.5$ shown in Table I. The computed approximation is then compared against the exact solution given by

$$y\left(t\right)=\left(t+1\right)^2-0.5e^t.$$

TABLE I. Absolute error comparison between the Runge-Kutta (RK) method of order 6 and the Euler-Maclaurin (EM) method of order 11 for Example 1 with step size $h = 0.2$

t_i	**RK Error** $\times 10^{-6}$	**EM Error** $\times 10^{-13}$
0.0	0.00000000	0.00000000
0.2	0.06348402	0.00222045
0.4	0.13430131	0.00444089
0.6	0.21258715	0.01554312
0.8	0.29817725	0.02220446
1.0	0.39046848	0.02664535
1.2	0.48823256	0.04884981
1.4	0.58936888	0.05329071
1.6	0.69057836	0.08881784
1.8	0.78693577	0.09769963
2.0	0.87133141	0.15099033

As observed, the Euler-Maclaurin Method (9) provides significantly more accurate approximations compared to the well-established Runge-Kutta method of order 6 shown in Table I. Fig. 1 and Fig. 2 illustrate a comparative analysis of the approximate solutions obtained by these methods, along with their respective absolute errors. To further validate the effectiveness of our approach, we extend our study to two additional examples.

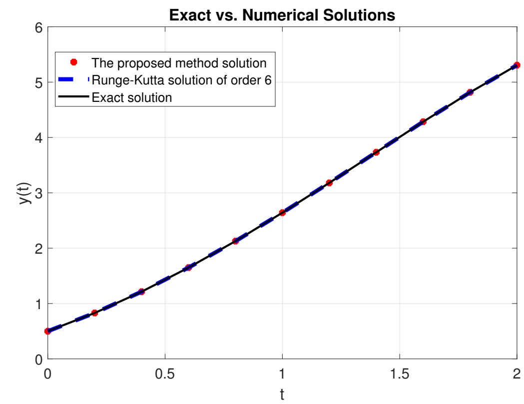

Fig. 1. Comparison of the exact solution with the Runge-Kutta (RK) method of order 6 and the Euler-Maclaurin (EM) method of order 11 for Example 1, using a step size of $h = 0.2$.

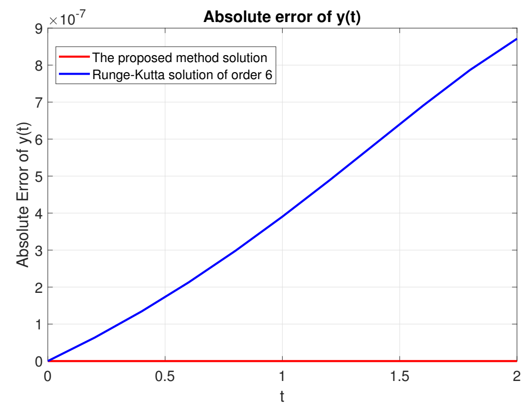

Fig. 2. Absolute errors of the Runge-Kutta (RK) method of order 6 and the Euler-Maclaurin (EM) method of order 11 for Example 1, using a step size of $h = 0.2$.

Example 2. The Euler-Maclaurin–Euler method (9) is applied to approximate the solution of the following initial-value problem:

$$y'\left(t\right)=\exp\left(t-y\right),\qquad 0\le t\le 1,\qquad y\left(0\right)=1. \tag{15}$$

The computations are performed using $N = 10$ subintervals, with a step size of $h = 0.1$, mesh points defined as $t_i = 0.1i$, and an initial value of $w_0 = 1$ shown in Table II. The obtained numerical solution is then compared against the exact solution given by

$$y\left(t\right)=\ln\left(\exp(t)+\exp(1)-1\right).$$

As observed, the Euler-Maclaurin method (9) provides significantly more accurate approximations compared to the well-known Runge-Kutta method of order 6. Fig. 3 and Fig. 4 illustrate the comparative performance of these methods, showcasing their respective absolute errors. To further refine our analysis and enhance the accuracy of the Euler-Maclaurin method (9), we now present the following example.

TABLE II. ABSOLUTE ERRORS IN THE RUNGE-KUTTA (RK) METHOD OF ORDER 6 AND EULER-MACLAURIN (EM) METHOD OF ORDER 11 FOR EXAMPLE 1 WITH STEP SIZE $h = 0.1$

t_i	RK Error $\times 10^{-9}$	EM Error $\times 10^{-15}$
0.0	0.00000000	0.00000000
0.1	0.01797051	0.22204460
0.2	0.03922263	0.22204460
0.3	0.06406808	0.22204460
0.4	0.09274892	0.44408921
0.5	0.12541213	0.44408921
0.6	0.16208479	0.44408921
0.7	0.20265589	0.22204460
0.8	0.24686364	0.44408921
0.9	0.29429104	0.22204460
1.0	0.34437453	0.22204460

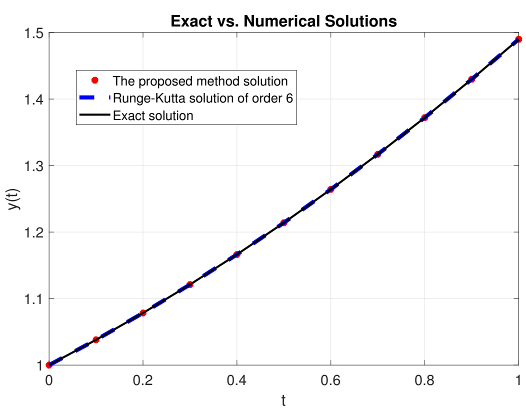

Fig. 3. Example 2: Comparison of the exact solution with the Euler-Maclaurin method (order 11) and the Runge-Kutta method (order 6), using step size $h = 0.1$

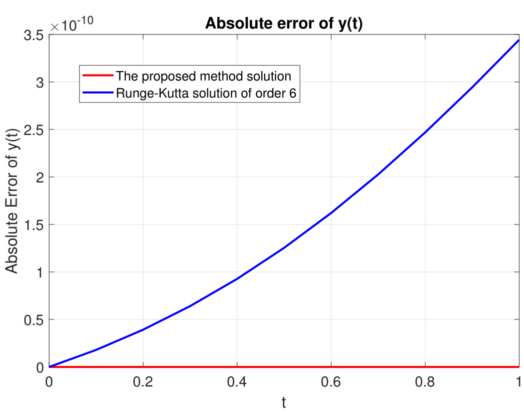

Fig. 4. Example 2: Absolute errors of the Euler-Maclaurin (order 11) and Runge-Kutta (order 6) methods, using step size $h = 0.1$

Example 3. The Euler-Maclaurin method (9) is applied to approximate the solution of the following system of linear initial-value problems:

$$\begin{cases} z_1'(s) = z_2, & z_1(0) = 1, \\ z_2'(s) = -z_1 - 2e^s + 1, & z_2(0) = 0, \\ z_3'(s) = -z_1 - e^s + 1, & z_3(0) = 1. \end{cases}$$

for $0 \leq s \leq 2$, with specific parameters set to $N = 10$, $h = 0.2$, and $t_i = 0.2i$. The computed approximation is then compared with the exact solution

$$\begin{cases} z_1(s) = \cos(s) + \sin(s) - e^s + 1, \\ z_2(s) = -\sin(s) + \cos(s) - e^s, \\ z_3(s) = -\sin(s) + \cos(s). \end{cases}$$

Furthermore, a comparative analysis is performed between the classical Runge-Kutta (RK) method and the proposed Euler-Maclaurin method. Specifically, Fig. 5, Fig. 7, and Fig. 9 illustrate the exact solution alongside the numerical approximations obtained using both the Euler-Maclaurin and RK methods with a step size of $h = 0.2$. Additionally, Fig. 6, Fig. 8, and Fig. 10, along with Tables III, IV, and V, present the absolute errors for both methods. The results indicate that the Euler-Maclaurin method consistently delivers highly accurate approximations, significantly outperforming the RK method.

TABLE III. ABSOLUTE ERRORS IN THE RUNGE-KUTTA (RK) METHOD OF ORDER 6 AND EULER-MACLAURIN (EM) METHOD OF ORDER 11 APPLIED IN EXAMPLE 1 WITH STEP SIZE $h = 0.2$

t_i	RK Error	EM Error $\times 10^{-15}$
0.0	0.00000000	0.00000000
0.2	0.11102230	0.01033334
0.4	0.22204460	0.02163267
0.6	0.11102230	0.03368738
0.8	0.11102230	0.04626921
1.0	0.00000000	0.05913556
1.2	0.22204460	0.07203310
1.4	0.11102230	0.08470149
1.6	0.19428903	0.09687735
1.8	0.24286129	0.10829828
2.0	0.11102230	0.11870699

TABLE IV. COMPARISON OF ABSOLUTE ERRORS BETWEEN THE RUNGE-KUTTA (RK) METHOD OF ORDER 6 AND THE EULER-MACLAURIN (EM) METHOD OF ORDER 11 APPLIED IN EXAMPLE 3 WITH STEP SIZE $h = 0.2$ FOR $z_2(s)$

t_i	RK Error	EM Error $\times 10^{-15}$
0.0	0.00000000	0.00000000
0.2	0.11102230	0.01033334
0.4	0.22204460	0.02163267
0.6	0.11102230	0.03368738
0.8	0.11102230	0.04626921
1.0	0.00000000	0.05913556
1.2	0.22204460	0.07203310
1.4	0.11102230	0.08470149
1.6	0.19428903	0.09687735
1.8	0.24286129	0.10829828
2.0	0.11102230	0.11870699

TABLE V. COMPARISON OF ABSOLUTE ERRORS BETWEEN THE RUNGE-KUTTA (RK) METHOD OF ORDER 6 AND THE EULER-MACLAURIN (EM) METHOD OF ORDER 11 APPLIED IN EXAMPLE 3 WITH STEP SIZE $h = 0.2$ FOR $z_3(s)$

t_i	**RK Error**	**EM Error** $\times 10^{-15}$
0.0	0.0000000	0.0000000
0.2	0.0027999	0.1110223
0.4	0.0234673	0.1110223
0.6	0.0650996	0.0000000
0.8	0.1301210	0.0000000
1.0	0.2202234	0.0555111
1.2	0.3363742	0.0555111
1.4	0.4788973	0.1526556
1.6	0.6476337	0.0173472
1.8	0.8421834	0.0277555
2.0	1.0622312	0.16653345

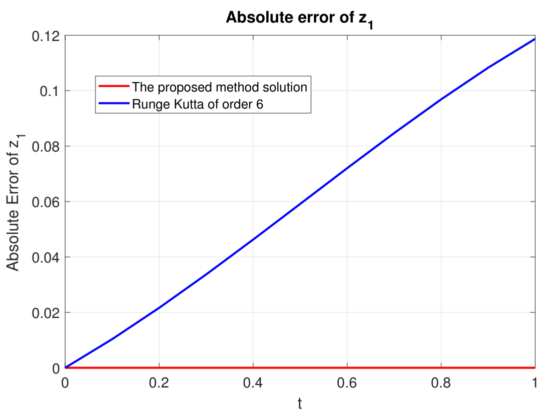

Fig. 5. Example 3: The exact solution of $z_1(s)$ compared with the Euler-Maclaurin (EM) and Runge-Kutta (RK) methods of order 11 and 6, respectively, using step size $h = 0.2$.

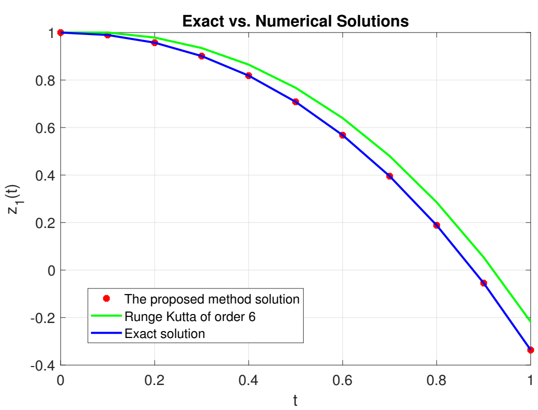

Fig. 6. Example 3: Absolute errors in $z_1(s)$ using the Euler-Maclaurin (EM) and Runge-Kutta (RK) methods of order 11 and 6, respectively, with step size $h = 0.2$.

VI. RECOMMENDATION

In this study, we have introduced a novel approach for approximating initial value problems (I.V.P.). Through a comprehensive analysis of method (8) and various numerical experiments, we have demonstrated that the Euler-Maclaurin method significantly outperforms well-established techniques, particularly the widely used Runge-Kutta method.

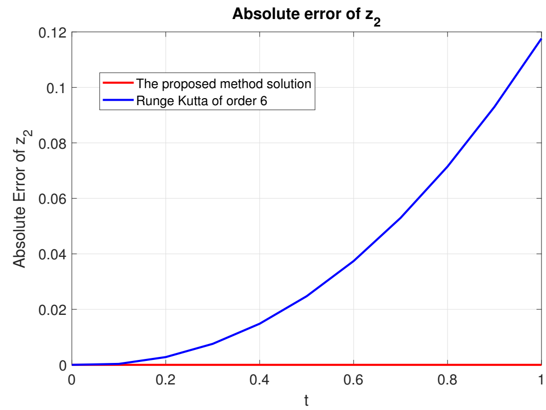

Fig. 7. Example 3: The exact solution of $z_2(s)$ compared with the Euler-Maclaurin (EM) and Runge-Kutta (RK) methods of order 11 and 6, respectively, with step size $h = 0.2$.

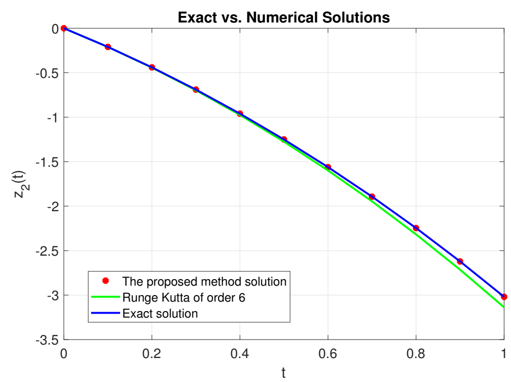

Fig. 8. Example 3: Absolute errors in $z_2(s)$ using the Euler-Maclaurin (EM) and Runge-Kutta (RK) methods of order 11 and 6, respectively, with step size $h = 0.2$.

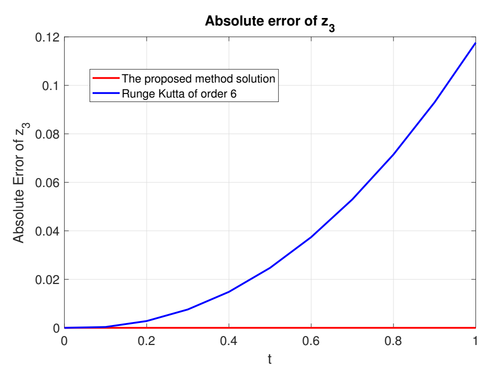

Fig. 9. Example 3: The exact solution of $z_3(s)$ compared with the Euler-Maclaurin (EM) and Runge-Kutta (RK) methods of order 11 and 6, respectively, with step size $h = 0.2$.

Our findings indicate that the Euler-Maclaurin method of order 11 exhibits superior accuracy compared to the Runge-Kutta method of order 6, especially in scenarios requiring precise analytic solutions. This advantage is evident in its ability to produce highly accurate approximations while maintaining a reduced absolute error.

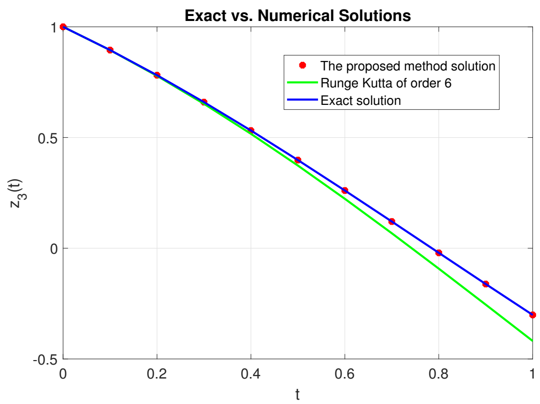

Fig. 10. Example 3: Absolute errors in $z_3(s)$ using the Euler-Maclaurin (EM) and Runge-Kutta (RK) methods of order 11 and 6, respectively, with step size $h = 0.2$.

Beyond its numerical superiority, the Euler-Maclaurin method showcases enhanced stability and faster convergence rates. The empirical results presented validate the method's robustness and efficiency across a wide range of mathematical modeling applications.

Furthermore, over extended computations, the Euler-Maclaurin method (8) of order $2n+1$ consistently demonstrates superior performance over the Runge-Kutta method, particularly when high-precision analytic solutions are needed. Additionally, its competitiveness across various scientific domains is reinforced by Example 3, where it effectively approximates a system of linear I.V.P., confirming its potential as a strong alternative to existing numerical methods.

VII. Conclusion and Future Works

In this study, we introduced an enhanced numerical method based on the Euler-Maclaurin formula for solving initial-value problems (I.V.P.), demonstrating its superior accuracy, stability, and convergence compared to the well-known Runge-Kutta methods. Numerical experiments confirmed its effectiveness, particularly for higher-order implementations. Future research could explore adaptive step-size techniques, applications to fractional and stochastic differential equations, and parallel computing implementations to further optimize performance and extend its applicability to real-world scientific and engineering problems.

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