Numerical and Analytical Investigations of Fractional Self-Adjoint Equations and Fractional Sturm-Liouville Problems via Modified Conformable Operator

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***Abstract*—This paper introduces a new modified conformable operator and explores its properties in detail. The motivation for studying this operator lies in its potential applications in fractional calculus and differential equations. We analyze the self-adjoint modified conformable equation by discussing the existence and uniqueness solution in the two cases, homogeneous and non-homogeneous, and establish its connection to specific modified conformable initial value problems. Furthermore, we investigate the modified conformable Sturm-Liouville problem by determining its eigenvalues and corresponding eigenfunctions. Key theoretical results related to orthogonality and linear dependence are presented. To validate the theoretical findings, we provide numerical methods and illustrative examples, demonstrating the applicability of our approach. These results contribute to a deeper understanding of modified conformable operators and their role in mathematical physics and engineering.**

Keywords*—Self Adjoint Equation; Sturm Liouville Problem; Eigenvalues; Eigenfunctions; Dependence; Orthogonality; Modified Conformable Operator; Initial Value Problem

I. INTRODUCTION

The concept of fractional derivatives dates back to the early development of calculus. In 1695, L'Hôpital inquired about the meaning of a derivative of order $n = \frac{1}{2}$, leading to centuries of research on fractional calculus [1]–[3]. Over time, various definitions have emerged, including the Riemann-Liouville, Caputo, Atangana-Baleanu, and Caputo-Fabrizio derivatives [4]–[6]. These formulations offer powerful tools for modeling complex systems with memory and hereditary properties. However, most classical fractional derivatives do not fully satisfy fundamental properties such as the product rule and chain rule, posing challenges in their application [7]–[11].

To address these limitations, a new local derivative known as the conformable fractional derivative was introduced [12]. This derivative is well-structured and preserves several fundamental properties of classical calculus.

Definition 1. *[12].* The conformable fractional derivative of a function of order is defined as:

$$T_\alpha(f)(x) = \lim_{\epsilon\to 0}\frac{f(x+\epsilon x^{1-\alpha}) - f(x)}{\epsilon}, \quad 0 < \alpha \leq 1, \quad x > 0.$$

This definition provides a simpler and more intuitive approach to fractional differentiation, making it a promising tool for various applications in applied mathematics and engineering.

Since its introduction, the conformable fractional derivative has gained significant attention, with researchers extending and generalizing its properties [13]–[17]. Khalil et al. [18]. provided a geometric interpretation of the conformable derivative, further strengthening its theoretical foundation. Its applications span numerous fields, including physics, engineering, and mathematical biology [19]–[24].

One important area where fractional derivatives play a crucial role is Sturm Liouville problems. These problems frequently arise in applied mathematics, particularly in solving separable linear partial differential equations. Sturm Liouville problems are essential in spectral theory, quantum mechanics, and vibration analysis [25]–[29]. Given the importance of these problems, extending them to fractional calculus using the modified conformable operator provides new insights and potential applications.

In this paper, we focus on a newly defined modified conformable operator and its applications in fractional differential equations. The main objectives of this study include a theoretical investigation where we explore both homogeneous and nonhomogeneous self-adjoint modified conformable differential equations of the form:

$$Lu(x) = 0, \quad \text{and} \quad Lu(x) = f(x),$$

where the self-adjoint modified conformable operator is defined as

$$Lu(x) = D^\alpha \left[p(x)\left(D^\alpha u(x) - \kappa_1(\alpha, x)u(x)\right)\right] + q(x)u(x),$$

with D^α representing the modified conformable derivative, and $p(x)$ and $q(x)$ being continuous functions. Moreover, the study extends to the modified conformable Sturm-Liouville problem of the form:

$$D^\alpha \left[p(x)\left(D^\alpha u(x) - \kappa_1(\alpha, x)u(x)\right)\right] + \left(\lambda r(x) + q(x)\right)u(x) = 0,$$

where λ and $u(x)$ denote the eigenvalues and eigenfunctions, respectively. Finally, we conclude our work by illustrating the numerical methods and simulations, where we develop and implement numerical techniques to approximate solutions and analyze their behavior through simulations.

II. Preliminaries

In this section, we examine a newly modified conformable derivative D^α of order α, where $0 < \alpha \leq 1$. Notably, D^0 corresponds to the identity operator, while D^1 reduces to the classical differential operator. It is worth noting that the contents of this section are adapted from [30].

Definition 2. Let $0 < \alpha \leq 1$, the operator D^α is said to be a modified conformable differential operator if and only if D^0 is the identity operator and D^1 is the classical differential operator. More formally, for a differentiable function $f(x)$, the modified conformable operator satisfies:

$$D^0 f(x) = f(x), \quad \text{and} \quad D^1 f(x) = \frac{d}{dx} f(x) = f'(x), \quad x \in \mathbb{R}. \tag{1}$$

Definition 3. Let $0 < \alpha \leq 1$, and consider two continuous functions $k_0, k_1 : [0,1] \times \mathbb{R} \to [0, \infty)$ satisfying the following conditions for all $x \in \mathbb{R}$:

$$\begin{aligned} &\lim_{\alpha\to 0^+} k_1(\alpha, x) = 1, && \lim_{\alpha\to 0^+} k_0(\alpha, x) = 0, \\ &\lim_{\alpha\to 1^-} k_1(\alpha, x) = 0, && \lim_{\alpha\to 1^-} k_0(\alpha, x) = 1, \\ &k_1(\alpha, x) \neq 0,\ \alpha \in [0,1), && k_0(\alpha, x) \neq 0,\ \alpha \in (0,1]. \end{aligned} \tag{2}$$

Given these conditions, the differential operator D^α is defined as:

$$D^\alpha f(x) = k_1(\alpha, x) f(x) + k_0(\alpha, x) f'(x), \tag{3}$$

where $f(x)$ is a differentiable function and $f'(x) = \frac{d}{dx} f(x)$. This operator is considered modified conformable under these assumptions.

The previous definition provides a generalized form of the given operator, allowing it to take on various forms. We refer to these variations as classes of the modified conformable derivative.

Example 1. The following are some classes of the modified conformable differential operator:

1) If we take $k_1(\alpha, x) = (1-\alpha)x^\alpha$ and $k_0(\alpha, x) = \alpha x^{1-\alpha}$ for any $x \in (0, \infty)$, the operator is expressed as follows:

$$D^\alpha f(x) = (1-\alpha)x^\alpha f(x) + \alpha x^{1-\alpha} f'(x),$$

which is modified conformable. This is because one can easily prove that the functions $k_1(\alpha, x)$ and $k_0(\alpha, x)$ satisfy (2), and the obtained operator verifies (1).

2) If we take $k_1(\alpha, x) = \cos\left(\frac{\alpha\pi}{2}\right) x^\alpha$ and $k_0(\alpha, x) = \sin\left(\frac{\alpha\pi}{2}\right) x^{1-\alpha}$ for any $x \in (0, \infty)$, then a similar class of the mentioned operator takes the following form:

$$D^\alpha f(x) = \cos\left(\frac{\alpha\pi}{2}\right) x^\alpha f(x) + \sin\left(\frac{\alpha\pi}{2}\right) x^{1-\alpha} f'(x).$$

Note that, unfortunately, $D^\alpha D^\beta \neq D^\beta D^\alpha$ in general.

Definition 4. Let $0 < \alpha \leq 1$, and consider the functions k_0 and k_1 defined on $[0,1] \times \mathbb{R}$ with values in $[0, \infty)$. Assume these functions are continuous and satisfy equation (2). Given a function $f(x, s)$ defined on $\mathbb{R}^2$ such that its derivative with respect to x, $\frac{d}{dx} f(x, s)$, exists for each fixed $s \in \mathbb{R}$, the partial modified conformable differential operator D_x^α is defined as

$$D_x^\alpha f(x, s) = k_1(\alpha, x) f(x, s) + k_0(\alpha, x) \frac{\partial}{\partial x} f(x, s). \tag{4}$$

This operator represents a generalized form of differentiation incorporating the functions k_0 and k_1, which depend on α and x.

Definition 5. Let $0 < \alpha \leq 1$, $s, x \in \mathbb{R}$ with $s \leq x$, and let the function $m : [s, x] \to \mathbb{R}$ be continuous. Let $k_0, k_1 : [0,1] \times \mathbb{R} \to [0, \infty)$ be continuous and satisfy (2) with m/k_0 and k_1/k_0

Riemann integrable on $[s,x]$. Then the modified conformable exponential function with respect to D^α is defined as

$$e_m(x,s) = e^{\int_s^x \frac{m(\lambda)-k_1(\alpha,\lambda)}{k_0(\alpha,\lambda)}d\lambda}, \quad e_0(x,s) = e^{\int_x^s \frac{k_1(\alpha,\lambda)}{k_0(\alpha,\lambda)}d\lambda}. \quad (5)$$

Now, based on (3) and (5), we conclude the following properties:

Lemma 1. Let the modified conformable differential operator D^α be given as in (3), where $0 < \alpha \leq 1$. Let the function $m : [s,x] \to \mathbb{R}$ be continuous, and the functions $k_0, k_1 : [0,1] \times \mathbb{R} \to [0,\infty)$ be continuous and satisfy (2) with $\frac{m}{k_0}$ and $\frac{k_1}{k_0}$ Riemann integrable on $[s,x]$. Assume the functions f and g are differentiable as needed. Then:

1) $D^\alpha[af + bg] = aD^\alpha[f] + bD^\alpha[g]$, for all $a, b \in \mathbb{R}$.
2) $D^\alpha c = ck_1(\alpha,x)$, for all constants $c \in \mathbb{R},\ x \in \mathbb{R}$.
3) $D^\alpha[fg] = fD^\alpha[g] + gD^\alpha[f] - fgk_1(\alpha,x)$, $x \in \mathbb{R}$.
4) $D^\alpha\left[\frac{f}{g}\right] = \frac{gD^\alpha[f]-fD^\alpha[g]}{g^2} + \frac{f}{g}k_1(\alpha,x)$, $x \in \mathbb{R}$ and $g \neq 0$.
5) For $\alpha \in (0,1]$ and fixed $s \in \mathbb{R}$, the exponential function satisfies
$$D_x^\alpha[e_m(x,s)] = m(x)e_m(x,s). \quad (6)$$
6) For $\alpha \in (0,1]$ and for the exponential function e_0 given in (5), we have
$$D^\alpha\left[\int_a^x \frac{f(s)e_0(x,s)}{k_0(\alpha,s)}ds\right] = f(x). \quad (7)$$

Proof. See [30].

Definition 6. Let $0 < \alpha \leq 1$ and $x_0 \in \mathbb{R}$. In light of (5) and Lemma 2.6 (v) and (vi), define the antiderivative via

$$\int D^\alpha f(x)d_\alpha x = f(x) + ce_0(x,x_0), \quad c \in \mathbb{R}.$$

In the same way, define the integral of f over the closed interval $[a,b]$ as follows:

$$\int_a^b f(s)e_0(x,s)d_\alpha s = \int_a^b \frac{f(s)e_0(x,s)}{k_0(\alpha,s)}ds, \quad d_\alpha s = \frac{ds}{k_0(\alpha,s)}. \quad (8)$$

Therefore, we can write:

$$e_0(x,s) = e^{\int_x^s \frac{k_1(\alpha,\lambda)}{k_0(\alpha,\lambda)}d\lambda} = e^{\int_x^s k_1(\alpha,\lambda)d_\alpha\lambda}.$$

The modified conformable integral has the following basic results.

Lemma 2. Let the conformable differential operator D^α be given as in (3) and the integral be given as (8) with $0 < \alpha \leq 1$. Let the functions $k_0, k_1 : [0,1] \times \mathbb{R} \to [0,\infty)$ be continuous and satisfy (2), and let f and g be differentiable as needed. Then:

1) The derivative of the definite integral of f is given by
$$D^\alpha\left[\int_a^x f(s)e_0(x,s)d_\alpha s\right] = f(x).$$
2) The definite integral of the derivative of f is given by
$$\int_a^x D^\alpha\left[f(s)e_0(x,s)\right]d_\alpha s = f(s)e_0(x,s)\Big|_{s=a}^{x} = f(x) - f(a)e_0(x,a).$$
3) An integration by parts formula is given as follows:
$$\int_a^b f(x)D^\alpha[g(x)]e_0(b,x)d_\alpha x = f(x)g(x)e_0(b,x)\Big|_{x=a}^{b} - \int_a^b g(x)\left(D^\alpha[f(x)] - k_1(\alpha,x)f(x)\right)e_0(b,x)d_\alpha x.$$
4) A version of the Leibniz rule for the differentiation of an integral is given by
$$D^\alpha\left[\int_a^x f(x,s)e_0(x,s)d_\alpha s\right] = f(x,x) + \int_a^x \left(D_x^\alpha[f(x,s)] - k_1(\alpha,x)f(x,s)\right)e_0(x,s)d_\alpha s.$$
If $e_0(x,s)$ is absent, then by (4) we have
$$D^\alpha\left[\int_a^x f(x,s)d_\alpha s\right] = \int_a^x D_x^\alpha f(x,s)d_\alpha s + f(x,x).$$

Proof. See [30].

One of the most important and essential theorems in our study is the Modified Conformable Constant Coefficients Theorem, which is presented below.

Theorem 1. *[31].*Let the functions $k_0, k_1 : [0,1] \times \mathbb{R} \to [0,\infty)$ be continuous and satisfy (2), and let D^α be as given in (3). Let $a, b, c \in \mathbb{R}$ be constants and $\alpha \in (0,1]$. Then the constant coefficients homogeneous modified conformable differential equation

$$aD^\alpha D^\alpha u(x) + bD^\alpha u(x) + cu(x) = 0, \quad x \in [x_0,\infty), \quad x_0 > 0, \quad (9)$$

has the associated characteristic equation

$$a\lambda^2 + b\lambda + c = 0, \quad (10)$$

and the general solution to (11) is given by one of the following cases:

- **Case 1:** If λ_1 and λ_2 are real distinct roots of (10), then
$$u(x) = c_1e_{\lambda_1}(x,x_0) + c_2e_{\lambda_2}(x,x_0),$$
- **Case 2:** If λ is a repeated root of (10), then
$$u(x) = c_1e_\lambda(x,x_0) + c_2e_\lambda(x,x_0)\int_{x_0}^x 1d_\alpha s,$$
- **Case 3:** If $\lambda = \zeta \pm i\beta$ is a complex root of (10), then
$$u(x) = c_1e_\zeta(x,x_0)\cos\left(\int_{x_0}^x \beta d_\alpha s\right) + c_2e_\zeta(x,x_0)\sin\left(\int_{x_0}^x \beta d_\alpha s\right),$$

where, in all cases, c_1 and c_2 are constants determined by the initial conditions.

Proof. We refer the reader to [31].

III. Main Results and Numerical Methods

This section presents the theoretical results and numerical methods for the modified conformable operator, focusing on the fractional self-adjoint equation and the associated Sturm-Liouville problem.

A. Fractional Self Adjoint Equation

We explore the fractional self-adjoint equation, highlighting its fundamental properties, solution behavior, and related analytical results.

1) Theoretical Investigation*:* This part investigates the fractional self-adjoint equation by deriving the characteristic equation and analyzing the solutions under different conditions.

Definition 7. The modified conformable homogeneous self-adjoint equation is given by

$$Lu(x) = D^{\alpha}\left[p(x)\left(D^{\alpha}u(x) - \kappa_1(\alpha,x)u(x)\right)\right] + q(x)u(x) = 0, \tag{11}$$

where $\alpha \in [0,1]$, D^{α} is defined as in (3), and $p(x), q(x)$ are continuous on $[x_0,\infty)$ with $p(x) \neq 0$ for all $x \in [x_0,\infty)$.

Theorem 2. A function $u(x) : [x_0,\infty) \to \mathbb{R}$ is a solution of (11) if $D^{\alpha}u(x)$ and $D^{\alpha}\left[p(x)\left(D^{\alpha}u(x) - \kappa_1(\alpha,x)u(x)\right)\right]$ are continuous on $[x_0,\infty)$ and satisfy $Lu(x) = 0$ for all $x \in [x_0,\infty)$.

In the following theorem, we establish the existence and uniqueness of solutions for the modified conformable non-homogeneous self-adjoint equation:

$$Lu(x) = f(x). \tag{12}$$

Theorem 3. (Existence and Uniqueness)
Assume that k_0, k_1 satisfy (3) and $\alpha \in [0,1]$, with given constants $u_0, u_1 \in \mathbb{R}$. If D^{α} is defined as in (3), and if p, q, f are continuous on $[x_0,\infty)$ such that $p(x) \neq 0$ for all $x \in [x_0,\infty)$, then the modified conformable initial value problem for the non-homogeneous self-adjoint equation:

$$Lu(x) = f(x), \quad u(x_0) = u_0, \quad D^{\alpha}u(x_0) = u_1 \tag{13}$$

has a unique solution on $[x_0,\infty)$.

Proof. First, we rewrite $Lu(x) = f(x)$ as an equivalent vector equation. Let $u(x)$ be a solution of $Lu(x) = f(x)$ and define

$$y(x) = p(x)\left(D^{\alpha}u(x) - k_1(\alpha,x)u(x)\right). \tag{14}$$

This gives

$$D^{\alpha}u(x) = k_1(\alpha,x)u(x) + \frac{y(x)}{p(x)}. \tag{15}$$

Since $u(x)$ is a solution of $Lu(x) = f(x)$ as defined in (11), we obtain:

$$D^{\alpha}y(x) = -q(x)u(x) + f(x). \tag{16}$$

Define the vector

$$z(x) = \begin{bmatrix} u(x) \\ y(x) \end{bmatrix}, \tag{17}$$

which transforms the equation into the modified conformable vector form:

$$D^{\alpha}z(x) = A(x)z(x) + b(x), \tag{18}$$

where

$$A(x) = \begin{bmatrix} k_1(\alpha,x) & \frac{1}{p(x)} \\ -q(x) & 0 \end{bmatrix}, \quad b(x) = \begin{bmatrix} 0 \\ f(x) \end{bmatrix}. \tag{19}$$

Using (3), we have

$$k_1(\alpha,x)z(x) + k_0(\alpha,x)z'(x) = A(x)z(x) + b(x) \tag{20}$$

which leads to

$$k_0(\alpha,x)z'(x) = -k_1(\alpha,x)z(x) + A(x)z(x) + b(x). \tag{21}$$

Hence, we obtain

$$z'(x) = \begin{bmatrix} 0 & \frac{1}{k_0(\alpha,x)p(x)} \\ \frac{-q(x)}{k_0(\alpha,x)} & \frac{-k_1(\alpha,x)}{k_0(\alpha,x)} \end{bmatrix} z(x) + \begin{bmatrix} 0 \\ \frac{f(x)}{k_0(\alpha,x)} \end{bmatrix}. \tag{22}$$

Since all functions involved are continuous, the existence and uniqueness follow from the classical ($\alpha = 1$) case.

The obtained solutions have some important properties, one of which is dependence, as shown below.

Corollary 1. (Dependence of Solutions)
Let $u_1(x), u_2(x)$ be two solutions of (11). Then

$$W(u_1,u_2)(x) = 0 \quad \forall x \in [x_0,\infty)$$

$\iff$ $u_1(x), u_2(x)$ are linearly dependent on $[x_0,\infty)$ or

$$W(u_1,u_2)(x) \neq 0 \quad \forall x \in [x_0,\infty)$$

$\iff$ $u_1(x), u_2(x)$ are linearly independent on $[x_0,\infty)$.

Proof. By Abel's Formula:

$$W(u_1,u_2)(x) = \frac{ce_0(x,x_0)}{p(x)}, \quad \forall x \in [x_0,\infty).$$

If $u_1(x)$ and $u_2(x)$ are linearly dependent, then clearly:

$$W(u_1,u_2)(x) = 0, \quad \forall x \in [x_0,\infty).$$

Conversely, if

$$W(u_1,u_2)(x) = 0, \quad \forall x \in [x_0,\infty),$$

then we have

$$\begin{aligned} &u_1(x)D^{\alpha}[u_2(x)] - u_2(x)D^{\alpha}[u_1(x)] \\ &\quad = u_1(x)\left(k_0(\alpha,x)u_2'(x) + k_1(\alpha,x)u_2(x)\right) \\ &\qquad - u_2(x)\left(k_0(\alpha,x)u_1'(x) + k_1(\alpha,x)u_1(x)\right) = 0. \end{aligned}$$

Simplifying, we obtain:

$$k_0(\alpha,x)\left(u_1(x)u_2'(x)-u_2(x)u_1'(x)\right)=0,$$

which implies that $u_1(x)$ and $u_2(x)$ are linearly dependent.

2) Numerical Method*:* To illustrate the obtained results on the fractional self-adjoint equation defined in (11), we assume that $k_0(\alpha,x), k_1(\alpha,x):[0,1]\times\mathbb{R}\to[0,\infty)$ are continuous and satisfy (2), such that $k_1(\alpha,x)$ is differentiable on $[0,\infty)$.
Let $\alpha\in[0,1]$. If we take

$$p(x)=e_0(x,0),\quad q(x)=e_0(x,0)+D^{\alpha}k_1(\alpha,x)e_0(x,0),$$

for $x\in[0,\infty)$, then by substituting p and q into (11), we obtain the following modified conformable initial value problem:

$$D^{2\alpha}u(x)+u(x)=0,\quad u(0)=2,\quad D^{\alpha}u(0)=5. \tag{23}$$

Now, since p and q are continuous with $p\neq 0$ on $[0,\infty)$, Theorem 3.3 ensures that the modified conformable initial value problem (23) has a unique solution. Applying Theorem 2.9 (Modified Conformable Constant Coefficients Theorem) shows that the associated characteristic equation takes the form:

$$\lambda^2+1=0,$$

which has two complex roots: $\lambda=\pm i$. Consequently, the unique solution to the given modified conformable initial value problem is given by:

$$u(x)=c_1e_0(x,0)\cos\left(\int_0^x 1d_{\alpha}s\right)+c_2e_0(x,0)\sin\left(\int_0^x 1d_{\alpha}s\right)$$

where c_1 and c_2 are constants to be determined based on the initial conditions. Using the initial conditions and the properties in Lemma 2.6 (iii) and Lemma 2.8 (iv), we find:

$$\begin{aligned}u(0)=&\left(c_1e_0(x,0)\cos\left(\int_0^x 1d_{\alpha}s\right)\right.\\&\left.+c_2e_0(x,0)\sin\left(\int_0^x 1d_{\alpha}s\right)\right)\Big|_{x=0}\\&=c_1e_0(0,0)\cos\left(\int_0^0 1d_{\alpha}s\right)+c_2e_0(0,0)\sin\left(\int_0^0 1d_{\alpha}s\right)\\&=c_1e_0(0,0)\cos(0)+c_2e_0(0,0)\sin(0)\\&=c_1e^{\int_0^0\frac{m(\lambda)-k_1(\alpha,\lambda)}{k_0(\alpha,\lambda)}d\lambda}=c_1=2.\end{aligned}$$

Also, we can have

$$\begin{aligned}D^{\alpha}u(0)=D^{\alpha}&\left[c_1e_0(x,0)\cos\left(\int_0^x 1d_{\alpha}s\right)\right.\\&\left.+c_2e_0(x,0)\sin\left(\int_0^x 1d_{\alpha}s\right)\right]\Big|_{x=0}\\&=c_1D^{\alpha}\left[e_0(x,0)\cos\left(\int_0^x 1d_{\alpha}s\right)\right]\Big|_{x=0}\\&\quad+c_2D^{\alpha}\left[e_0(x,0)\sin\left(\int_0^x 1d_{\alpha}s\right)\right]\Big|_{x=0}\end{aligned}$$

or

$$\begin{aligned}D^{\alpha}u(0)=c_1&\left[e_0(x,0)D^{\alpha}\left(\cos\left(\int_0^x 1d_{\alpha}s\right)\right)\right.\\&\left.+\cos\left(\int_0^x 1d_{\alpha}s\right)D^{\alpha}\left(e_0(x,0)\right)\right]\Big|_{x=0}\\&-c_1\left[e_0(x,0)\cos\left(\int_0^x 1d_{\alpha}s\right)k_1(\alpha,x)\right]\Big|_{x=0}\\&+c_2\left[e_0(x,0)D^{\alpha}\left(\sin\left(\int_0^x 1d_{\alpha}s\right)\right)\right.\\&\left.+\sin\left(\int_0^x 1d_{\alpha}s\right)D^{\alpha}\left(e_0(x,0)\right)\right]\Big|_{x=0}\\&-c_2\left[e_0(x,0)\sin\left(\int_0^x 1d_{\alpha}s\right)k_1(\alpha,x)\right]\Big|_{x=0}\\&=c_1\left[k_1(\alpha,0)+1-k_1(\alpha,0)\right]+c_2\left[1+0-0\right]\\&=c_1+c_2=5.\end{aligned}$$

Hence, we have

$$c_2=3.$$

Consequently, we conclude that the solution of the newly modified conformable self-adjoint equation, subject to its initial conditions, is given by:

$$u(x)=2e_1(x,2)\cos\left(\int_2^x 1d_{\alpha}s\right)+3e_1(x,2)\sin\left(\int_2^x 1d_{\alpha}s\right).$$

3) Simulation*:* To clarify the solution of the fractional initial value problem (23), analyze its behavior, and examine the effect of the fractional order α, we will use the Mathematica program to approximate and visualize the solution. First, the obtained solution $u(x)$ is presented in its general form as follows:

$$u(x)=2e_1(x,2)\cos\left(\int_2^x 1d_{\alpha}s\right)+3e_1(x,2)\sin\left(\int_2^x 1d_{\alpha}s\right).$$

Since

$$e_m(x,s)=e^{\int_s^x\frac{m(\lambda)-k_1(\alpha,\lambda)}{k_0(\alpha,\lambda)}d\lambda},\quad\text{and}\quad d_{\alpha}s=\frac{ds}{k_0(\alpha,s)}.$$

If we take $k_1(\alpha,x)=(1-\alpha)x^{\alpha}$ and $k_0(\alpha,x)=\alpha x^{1-\alpha}$, we will obtain the following class of the modified conformable differential operator:

$$D^{\alpha}f(x)=(1-\alpha)x^{\alpha}f(x)+\alpha x^{1-\alpha}f'(x).$$

Consequently,

$$e_1(x,2)=e^{\int_2^x\frac{1-(1-\alpha)\lambda^{\alpha}}{\alpha\lambda^{(1-\alpha)}}d\lambda},\quad\text{and}\quad d_{\alpha}s=\frac{1}{\alpha\lambda^{(1-\alpha)}}d\lambda.$$

Hence, the solution becomes:

$$\begin{aligned}u(x)=2e^{\int_2^x\frac{1-(1-\alpha)\lambda^{\alpha}}{\alpha\lambda^{(1-\alpha)}}d\lambda}&\cos\left(\int_2^x\frac{1}{\alpha\lambda^{(1-\alpha)}}d\lambda\right)\\+3e^{\int_2^x\frac{1-(1-\alpha)\lambda^{\alpha}}{\alpha\lambda^{(1-\alpha)}}d\lambda}&\sin\left(\int_2^x\frac{1}{\alpha\lambda^{(1-\alpha)}}d\lambda\right).\end{aligned}$$

Using the Mathematica program, we plot the obtained solution for different values of the fractional order α, resulting in the following content.

Discussion:
After a careful comparative study between the presented figures, we conclude the following points:

- The first graph (Fig. 1) represents the solution obtained using the usual derivative, which demonstrates greater stability compared to the fractional derivative.
- The fractional order α significantly influences the solution's behavior, as shown in Fig. 2, Fig. 3, and Fig. 4.
- As α increases, the solution becomes more stable.
- Conversely, smaller values of α introduce more chaotic behavior in the solution.
- These findings highlight the role of fractional derivatives in modeling dynamic systems with varying degrees of stability.

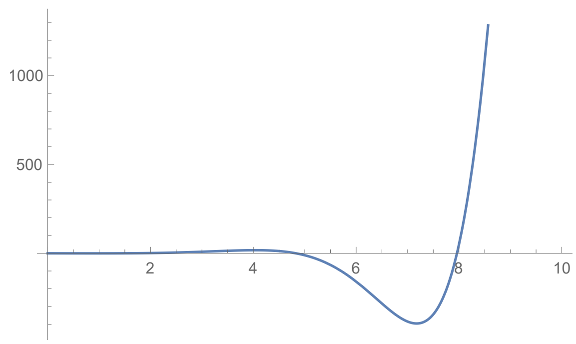

Fig. 1. The solution $u(x)$ with $\alpha = 1$

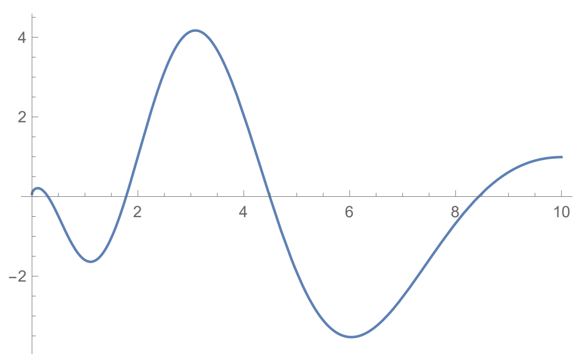

Fig. 2. The solution $u(x)$ with $\alpha = \frac{1}{2}$

B. Fractional Sturm-Liouville Equation

1) Conceptual Analysis: This section introduces the fractional Sturm-Liouville equation, which generalizes the classical problem using the modified conformable derivative.

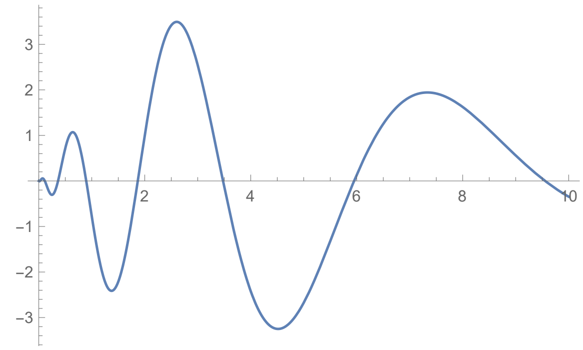

Fig. 3. The solution $u(x)$ with $\alpha = \frac{1}{4}$

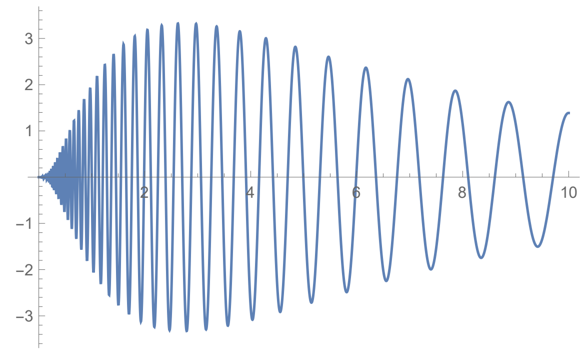

Fig. 4. The solution $u(x)$ with $\alpha = \frac{1}{50}$

Definition 8. The modified conformable Sturm-Liouville equation is given by

$$D^{\alpha}\left[p(x)\left(D^{\alpha}u(x) - \kappa_1(\alpha, x)u(x)\right)\right] + (\lambda r(x) + q(x))u(x) = 0, \tag{24}$$

where $p(x)$, $q(x)$, and $r(x)$ are real continuous functions on $[x_0, \infty)$, and

$$p(x) \neq 0, \quad x \in [x_0, \infty),$$

with $r(x) \geq 0$ being non-trivially zero on $[x_0, \infty)$.
Equation (24) can be expressed in the form

$$Lu(x) = -\lambda r(x)u(x),$$

where L is defined as

$$Lu(x) = D^{\alpha}\left[p(x)\left(D^{\alpha}u(x) - \kappa_1(\alpha, x)u(x)\right)\right] + q(x)u(x). \tag{25}$$

Definition 9. Let a_1, a_2, b_1, b_2 be real constants satisfying

$$a_1^2 + a_2^2 > 0,$$
$$b_1^2 + b_2^2 > 0.$$

Then, the modified conformable Sturm-Liouville problem is given by

$$\begin{aligned} &Lu(x) = -\lambda r(x)u(x), \\ &a_1 u(a) + a_2 D^{\alpha}u(a) = 0, \\ &b_1 u(b) + b_2 D^{\alpha}u(b) = 0, \end{aligned} \tag{26}$$

where $a_1, a_2, b_1, b_2 \in \mathbb{R}$.

Definition 10. (Eigenvalue)
The number λ is called an eigenvalue for the modified conformable Sturm-Liouville problem (26) if the problem (26), with such λ, has a nontrivial solution $u(x)$, which is referred to as the eigenfunction corresponding to the eigenvalue λ.

Theorem 4. Let λ be an eigenvalue of the modified conformable Sturm-Liouville problem (26). Then $\lambda \in \mathbb{R}$, i.e., all eigenvalues of (26) are real.

Proof. Let λ be an eigenvalue of (26) and $\varphi(x)$ be the corresponding eigenfunction. Then,

$$L\varphi(x) = -\lambda r(x)\varphi(x). \tag{27}$$

Taking the conjugate of both sides and multiplying by $e_0(x,b)\varphi(x)$, we obtain

$$L\overline{\varphi(x)}e_0(x,b)\varphi(x) = -\bar{\lambda}r(x)\overline{\varphi(x)}e_0(x,b)\varphi(x). \tag{28}$$

From (28), we deduce

$$L\overline{\varphi(x)}e_0(x,b)\varphi(x) = -\bar{\lambda}r(x)e_0(x,b)|\varphi(x)|^2. \tag{29}$$

By integrating both sides of (29) over the interval $[a,b]$, we get

$$\int_a^b L\overline{\varphi(x)}e_0(x,b)\varphi(x)d_\alpha x = \int_a^b -\bar{\lambda}r(x)e_0(x,b)|\varphi(x)|^2 d_\alpha x.$$

Similarly, multiplying both sides of (27) by $e_0(b,x)\overline{\varphi(x)}$ and integrating gives

$$\int_a^b L\varphi(x)e_0(x,b)\overline{\varphi(x)}d_\alpha x = \int_a^b -\lambda r(x)e_0(x,b)|\varphi(x)|^2 d_\alpha x.$$

Since L is self-adjoint, we have

$$\int_a^b L\varphi(x)e_0(x,b)\overline{\varphi(x)}d_\alpha x = \int_a^b L\overline{\varphi(x)}e_0(x,b)\varphi(x)d_\alpha x.$$

Substituting (29) into the above equation, we obtain

$$\int_a^b -\lambda r(x)|\varphi(x)|^2 e_0(x,b)d_\alpha x = \int_a^b -\bar{\lambda}r(x)e_0(x,b)|\varphi(x)|^2 d_\alpha x,$$

which implies

$$\lambda = \bar{\lambda}.$$

Hence, all eigenvalues are real, which completes the proof.

Theorem 5. (Dependence of Eigenfunctions)
All the eigenvalues of (26) are simple. Equivalently, if λ is an eigenvalue of (26) and $\varphi_1(x), \varphi_2(x)$ are the corresponding eigenfunctions of λ, then $\varphi_1(x)$ and $\varphi_2(x)$ are linearly dependent.

Proof. Applying the boundary conditions of (26), we have

$$a_1\varphi_1(a) + a_2 D^\alpha \varphi_1(a) = 0,$$
$$a_1\varphi_2(a) + a_2 D^\alpha \varphi_2(a) = 0.$$

Next, we calculate the value of the modified conformable Wronskian of $\varphi_1(x)$ and $\varphi_2(x)$ at $x = a$:

$$\begin{aligned} W(\varphi_1,\varphi_2)(a) &= \varphi_1(a)D^\alpha\varphi_2(a) - \varphi_2(a)D^\alpha\varphi_1(a), \\ &= \varphi_1(a)\left(\frac{-a_1\varphi_2(a)}{a_2}\right) - \varphi_2(a)\left(\frac{-a_1\varphi_1(a)}{a_2}\right), \\ &= 0. \end{aligned}$$

By Corollary 3.4, we conclude that $\varphi_1(x)$ and $\varphi_2(x)$ are linearly dependent. Hence, all eigenvalues of (26) are simple.

Theorem 6. (Orthogonality of Eigenfunctions)
Eigenfunctions corresponding to distinct eigenvalues are orthogonal.

Proof. Let λ_1 and λ_2 be two distinct eigenvalues of (26) with corresponding eigenfunctions $\varphi_1(x)$ and $\varphi_2(x)$, respectively. Then, by (26), we have

$$L\varphi_1(x) + \lambda_1 r(x)\varphi_1(x) = 0, \tag{30}$$

$$L\varphi_2(x) + \lambda_2 r(x)\varphi_2(x) = 0. \tag{31}$$

Multiplying both sides of (30) by $e_0(x,b)\varphi_2(x)$ and both sides of (31) by $e_0(x,b)\varphi_1(x)$, we get

$$L\varphi_1(x)e_0(x,b)\varphi_2(x) + \lambda_1 r(x)\varphi_1(x)e_0(x,b)\varphi_2(x) = 0, \tag{32}$$

$$L\varphi_2(x)e_0(x,b)\varphi_1(x) + \lambda_2 r(x)\varphi_2(x)e_0(x,b)\varphi_1(x) = 0. \tag{33}$$

Now, subtracting (32) from (33) and integrating both sides over $[a,b]$, we obtain

$$\int_a^b \left(L\varphi_2(x)e_0(x,b)\varphi_1(x) - L\varphi_1(x)e_0(x,b)\varphi_2(x)\right)d_\alpha x - \int_a^b (\lambda_1 - \lambda_2)\, r(x)\varphi_1(x)e_0(x,b)\varphi_2(x)d_\alpha x = 0.$$

Since L is self-adjoint, we have

$$-\int_a^b (\lambda_1 - \lambda_2)\, r(x)\varphi_1(x)e_0(x,b)\varphi_2(x)d_\alpha x = 0.$$

As $\lambda_1 \neq \lambda_2$, this implies

$$\int_a^b r(x)\varphi_1(x)e_0(x,b)\varphi_2(x)d_\alpha x = 0.$$

This establishes the orthogonality of $\varphi_1(x)$ and $\varphi_2(x)$.

2) Numerical Approaches: To apply the results obtained for the fractional Sturm-Liouville problem, we introduce two problems discussed in the subsequent content.

Problem 1. Let $\kappa_0(\alpha, x), \kappa_1(\alpha, x) : [0, 1] \times \mathbb{R} \to [0, \infty)$ be continuous and satisfy (2), where $\kappa_1(\alpha, x)$ is a real constant. We aim to determine the eigenvalues and the corresponding eigenfunctions for the following modified conformable Sturm-Liouville problem:

$$D^{2\alpha}u(x) - 4lD^{\alpha}u(x) + \lambda u(x) = 0, \quad u(0) = u(2l) = 0, \tag{34}$$

where l is a positive constant. First, we will derive the characteristic equation for the above problem:

$$m^2 - 4lm + \lambda = 0,$$

which has the following roots:

$$m = 2l \pm \sqrt{4l^2 - \lambda}.$$

The Modified Conformable Constant Coefficients Theorem (Theorem 2.9) discusses the following cases:
Case 1: If $4l^2 - \lambda > 0$, then we have real distinct roots given by:

$$m = 2l \pm \sqrt{4l^2 - \lambda},$$

and by the constant coefficients modified conformable equation theorem, the solution is given by:

$$u(x) = c_1 e_{2l+\sqrt{4l^2-\lambda}}(x, 0) + c_2 e_{2l-\sqrt{4l^2-\lambda}}(x, 0),$$

where c_1 and c_2 are constants. Using the boundary conditions, we obtain:

$$\begin{aligned} u(0) &= \left(c_1 e_{2l+\sqrt{4l^2-\lambda}}(x, 0) + c_2 e_{2l-\sqrt{4l^2-\lambda}}(x, 0)\right)\Big|_{x=0} \\ &= c_1 e_{2l+\sqrt{4l^2-\lambda}}(0, 0) + c_2 e_{2l-\sqrt{4l^2-\lambda}}(0, 0) \\ &= c_1 + c_2 = 0. \end{aligned}$$

This implies that $c_2 = -c_1$, and thus:

$$\begin{aligned} u(2l) &= \left(c_1 e_{2l+\sqrt{4l^2-\lambda}}(x, 0) + c_2 e_{2l-\sqrt{4l^2-\lambda}}(x, 0)\right)\Big|_{x=2l} \\ &= \left(c_1 e_{2l+\sqrt{4l^2-\lambda}}(x, 0) - c_1 e_{2l-\sqrt{4l^2-\lambda}}(x, 0)\right)\Big|_{x=2l} \\ &= c_1\left(e_{2l+\sqrt{4l^2-\lambda}}(2l, 0) - e_{2l-\sqrt{4l^2-\lambda}}(2l, 0)\right) \\ &= 0. \end{aligned}$$

Since:

$$\left(e_{2l+\sqrt{4l^2-\lambda}}(2l, 0) - e_{2l-\sqrt{4l^2-\lambda}}(2l, 0)\right) \neq 0,$$

we conclude that $c_1 = c_2 = 0$, implying that λ in this case is not an eigenvalue of the problem.
Case 2: If $4l^2 - \lambda = 0$, then we have real repeated roots $m = 2l$, so by the constant coefficients modified conformable equation theorem, the solution is given by:

$$u(x) = c_1 e_{2l}(x, 0) + c_2 e_{2l}(x, 0) \int_0^x 1 d_\alpha s.$$

Now, using the boundary conditions, we get:

$$\begin{aligned} u(0) &= \left(c_1 e_{2l}(x, 0) + c_2 e_{2l}(x, 0) \int_0^x 1 d_\alpha s\right)\Big|_{x=0} \\ &= c_1 e_{2l}(0, 0) + c_2 e_{2l}(0, 0) \int_0^0 1 d_\alpha s \\ &= c_1 = 0. \end{aligned}$$

Similarly,

$$\begin{aligned} u(2l) &= \left(c_1 e_{2l}(x, 0) + c_2 e_{2l}(x, 0) \int_0^x 1 d_\alpha s\right)\Big|_{x=2l} \\ &= c_2 e_{2l}(2l, 0) \int_0^{2l} 1 d_\alpha s \\ &= 0. \end{aligned}$$

It is obvious that:

$$e_{2l}(2l, 0) \int_0^{2l} 1 d_\alpha s \neq 0.$$

Therefore, we conclude that $c_1 = c_2 = 0$, implying that λ in this case is not an eigenvalue of the problem.
Case 3: If $4l^2 - \lambda < 0$, then we have complex roots $m = 2l \pm i\sqrt{\lambda - 4l^2}$, so by the constant coefficients modified conformable equation theorem, the solution is given by:

$$\begin{aligned} u(x) = c_1 e_{2l}(x, 0) \cos\left(\int_0^x \sqrt{\lambda - 4l^2} d_\alpha s\right) \\ + c_2 e_{2l}(x, 0) \sin\left(\int_0^x \sqrt{\lambda - 4l^2} d_\alpha s\right). \end{aligned}$$

Now, using the boundary condition:

$$\begin{aligned} u(0) &= \Bigg(c_1 e_{2l}(x, 0) \cos\left(\int_0^x \sqrt{\lambda - 4l^2} d_\alpha s\right) \\ &\quad + c_2 e_{2l}(x, 0) \sin\left(\int_0^x \sqrt{\lambda - 4l^2} d_\alpha s\right)\Bigg)\Bigg|_{x=0} \\ &= \Bigg(c_1 e_{2l}(0, 0) \cos\left(\int_0^0 \sqrt{\lambda - 4l^2} d_\alpha s\right) \\ &\quad + c_2 e_{2l}(0, 0) \sin\left(\int_0^0 \sqrt{\lambda - 4l^2} d_\alpha s\right)\Bigg) \\ &= c_1 = 0. \end{aligned}$$

Similarly,

$$\begin{aligned} u(2l) &= \Bigg(c_1 e_{2l}(x, 0) \cos\left(\int_0^x \sqrt{\lambda - 4l^2} d_\alpha s\right) \\ &\quad + c_2 e_{2l}(x, 0) \sin\left(\int_0^x \sqrt{\lambda - 4l^2} d_\alpha s\right)\Bigg)\Bigg|_{x=2l} \\ &= c_2 e_{2l}(2l, 0) \sin\left(\int_0^{2l} \sqrt{\lambda - 4l^2} d_\alpha s\right) = 0. \end{aligned}$$

Then, we get

$$c_2 \sin\left(\int_0^{2l} \sqrt{\lambda - 4l^2} d_\alpha s\right) = 0.$$

Assume that $c_2 \neq 0$, we get:

$$\sin\left(\int_0^{2l}\sqrt{\lambda-4l^2}d_\alpha s\right)=0.$$

Therefore, we have

$$\int_0^{2l}\sqrt{\lambda-4l^2}d_\alpha s=n\pi,\quad n\in\mathbb{N}.$$

Finally, we conclude that:

$$\lambda_n=4l^2+\left(\frac{n\pi}{\int_0^{2l}1d_\alpha s}\right)^2,\quad n\in\mathbb{N}.$$

The eigenfunctions corresponding to the eigenvalues λ_n are given by:

$$u_n(x)=e_{2l}(x,0)\sin\left(\frac{n\pi\int_0^x 1d_\alpha s}{\int_0^{2l}1d_\alpha s}\right),\quad n\in\mathbb{N}.$$

Simulation:

To better understand the solution of Problem 1, analyze its behavior, and assess the impact of the fractional order α, we will employ the Mathematica program to approximate the obtained eigenvalues and their corresponding eigenfunctions. To develop and implement the numerical techniques to approximate solutions and illustrate their behavior through simulations, we need to consider some classes of the modified conformable differential operator. For this purpose, we will choose the two obtained classes in Example 2.3.

Class 1: If we take $k_1(\alpha,x)=(1-\alpha)x^\alpha$ and $k_0(\alpha,x)=\alpha x^{1-\alpha}$, we will obtain the following class of the modified conformable differential operator:

$$D^\alpha f(x)=(1-\alpha)x^\alpha f(x)+\alpha x^{1-\alpha}f'(x).\tag{35}$$

Consequently,

$$e_{2l}(x,0)=e^{\int_0^x\frac{2l-(1-\alpha)\lambda^\alpha}{\alpha\lambda^{(1-\alpha)}}d\lambda},\quad\text{and}\quad d_\alpha s=\frac{1}{\alpha\lambda^{(1-\alpha)}}d\lambda.\tag{36}$$

As a result, the solution is expressed as:

$$\lambda_n=4l^2+\left(\frac{n\pi}{\int_0^{2l}\frac{1}{\alpha\lambda^{(1-\alpha)}}d\lambda}\right)^2,\quad n\in\mathbb{N}.\tag{37}$$

$$u_n(x)=e^{\int_0^x\frac{2l-(1-\alpha)\lambda^\alpha}{\alpha\lambda^{(1-\alpha)}}d\lambda}\times\sin\left(\frac{n\pi\int_0^x\frac{1}{\alpha\lambda^{(1-\alpha)}}d\lambda}{\int_0^{2l}\frac{1}{\alpha\lambda^{(1-\alpha)}}d\lambda}\right),\quad n\in\mathbb{N}.\tag{38}$$

For a fixed $l=1$, and different values of the fractional order α and the integer n, we obtain the subsequent results for the eigenvalues (λ_n) and their corresponding eigenfunctions $(u_n(x))$. Here, $RA(\lambda_n)$ denotes the root approximant and $NV(\lambda_n)$ represents the numerical value with 10 digits, as shown in Table I.

For a clearer visualization of the eigenfunctions $(u_n(x))$, a graphical representation can be provided by fixing the fractional order α and varying the integer n, as illustrated in Fig. 5–7.

Class 2: If we take $k_1(\alpha,x)=\cos\left(\frac{\alpha\pi}{2}\right)x^\alpha$ and $k_0(\alpha,x)=\sin\left(\frac{\alpha\pi}{2}\right)x^{1-\alpha}$ for any $x\in(0,\infty)$, then a similar class of the mentioned operator takes the following form:

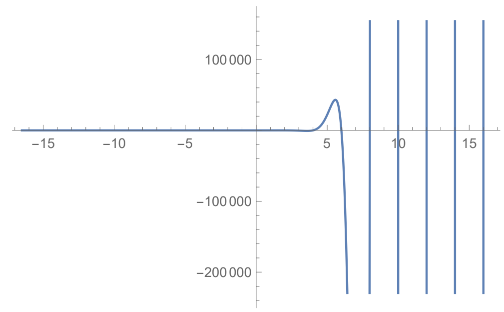

Fig. 5. $u_n(x)$ with $\alpha=1$ and $n=1$

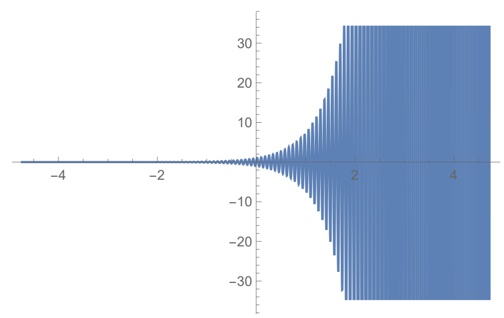

Fig. 6. $u_n(x)$ with $\alpha=1$ and $n=50$.

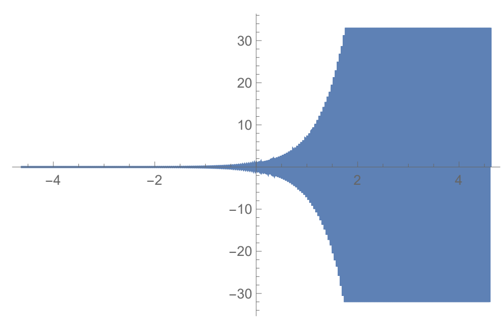

Fig. 7. $u_n(x)$ with $\alpha=1$ and $n=100$.

$$D^\alpha f(x)=\cos\left(\frac{\alpha\pi}{2}\right)x^\alpha f(x)+\sin\left(\frac{\alpha\pi}{2}\right)x^{1-\alpha}f'(x).\tag{39}$$

Consequently, the modified conformable exponential function becomes:

$$e_{2l}(x,0)=e^{\int_0^x\frac{2l-\cos\left(\frac{\alpha\pi}{2}\right)\lambda^\alpha}{\sin\left(\frac{\alpha\pi}{2}\right)\lambda^{1-\alpha}}d\lambda}\text{ and }d_\alpha s=\frac{1}{\sin\left(\frac{\alpha\pi}{2}\right)\lambda^{1-\alpha}}d\lambda.\tag{40}$$

TABLE I. EIGENVALUES (λ_n) AND THEIR CORRESPONDING EIGENFUNCTIONS ($u_n(x)$) FOR DIFFERENT VALUES OF α AND n WITH ROOT APPROXIMANT ($RA(\lambda_n)$) AND NUMERICAL VALUE ($NV(\lambda_n)$) TO 10 DIGITS

α	n	$RA(\lambda_n)$	$NV(\lambda_n)$	$u_n(x)$
1	1	6.47	6.467401100	$e^{2x}\sin\left(\frac{\pi x}{2}\right)$
1	50	$\frac{1117223}{181}$	6172.502751	$e^{2x}\sin(25\pi x)$
1	100	$\frac{2221021}{90}$	24678.01100	$e^{2x}\sin(50\pi x)$
$\frac{1}{2}$	1	4.31	4.308425138	$e^{(8\sqrt{x}-x)}\sin\left(\frac{\pi\sqrt{x}}{\sqrt{2}}\right)$
$\frac{1}{2}$	50	$\frac{1}{22}\left(8633+\sqrt{70869165}\right)$	775.0628438	$e^{(8\sqrt{x}-x)}\sin\left(25\sqrt{2}\pi\sqrt{x}\right)$
$\frac{1}{2}$	100	$\frac{1}{8}\left(12345+63\sqrt{38497}\right)$	3088.251375	$e^{(8\sqrt{x}-x)}\sin\left(50\sqrt{2}\pi\sqrt{x}\right)$
$\frac{1}{20}$	1	$\frac{69497}{17374}$	4.000057554	$e^{\left(800x^{\frac{1}{20}}-190x^{\frac{1}{10}}\right)}\sin\left(\frac{\pi x^{\frac{1}{20}}}{2^{\frac{1}{20}}}\right)$
$\frac{1}{20}$	50	4.14	4.143885414	$e^{\left(800x^{\frac{1}{20}}-190x^{\frac{1}{10}}\right)}\sin\left(25\times 2^{\frac{19}{20}}\pi x^{\frac{1}{20}}\right)$
$\frac{1}{20}$	100	4.58	4.575541657	$e^{\left(800x^{\frac{1}{20}}-190x^{\frac{1}{10}}\right)}\sin\left(50\times 2^{\frac{19}{20}}\pi x^{\frac{1}{20}}\right)$

As a result, the eigenvalues λ_n are expressed as:

$$\lambda_n = 4l^2 + \left(\frac{n\pi}{\int_0^{2l}\frac{1}{\sin\left(\frac{\alpha\pi}{2}\right)\lambda^{1-\alpha}}d\lambda}\right)^2, \quad n\in\mathbb{N}. \tag{41}$$

The corresponding eigenfunctions $u_n(x)$ are given by:

$$u_n(x) = e^{\int_0^x \frac{2l-\cos\left(\frac{\alpha\pi}{2}\right)\lambda^{\alpha}}{\sin\left(\frac{\alpha\pi}{2}\right)\lambda^{1-\alpha}}d\lambda}\times\sin\left(\frac{n\pi\int_0^x\frac{1}{\sin\left(\frac{\alpha\pi}{2}\right)\lambda^{1-\alpha}}d\lambda}{\int_0^{2l}\frac{1}{\sin\left(\frac{\alpha\pi}{2}\right)\lambda^{1-\alpha}}d\lambda}\right), \tag{42}$$

for $n\in\mathbb{N}$.

For a fixed $l = 1$, varying the fractional order α and the integer n yields the following results for the eigenvalues (λ_n) and their corresponding eigenfunctions ($u_n(x)$). In Table II, $RA(\lambda_n)$ denotes the root approximant and $NV(\lambda_n)$ represents the numerical value with 10 digits.

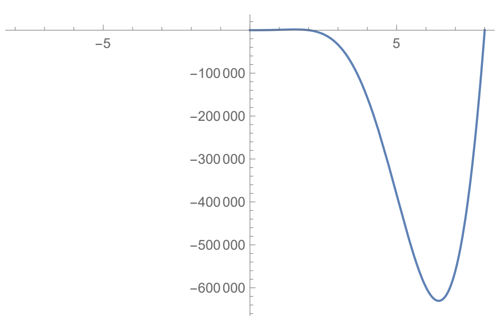

Fig. 8. $u_n(x)$ with $\alpha = \frac{1}{2}$ and $n = 1$

To better visualize the eigenfunctions $u_n(x)$, a graphical representation can be shown by keeping the fractional order α fixed while varying the integer n, as demonstrated in Figs. 14–22.

Comparative Analysis:

After an in-depth analysis of the problem using the Mathematica program, along with a careful comparison of the eigenvalues, their corresponding eigenfunctions, and the obtained graphs employing various classes of modified conformable differential operators, we arrive at the following conclusions:

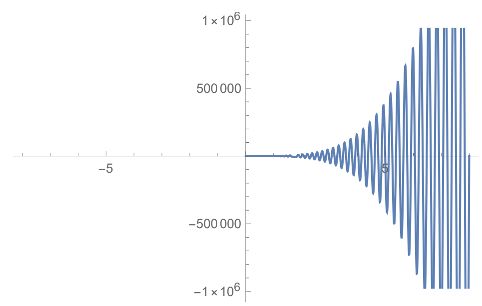

Fig. 9. $u_n(x)$ with $\alpha = \frac{1}{2}$ and $n = 50$

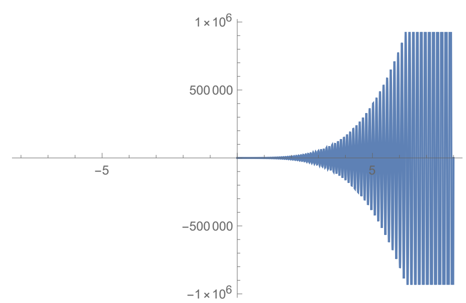

Fig. 10. $u_n(x)$ with $\alpha = \frac{1}{2}$ and $n = 100$.

- Both classes used (Class 1: $D^{\alpha}f(x) = (1-\alpha)x^{\alpha}f(x) + \alpha x^{1-\alpha}(\alpha, x)f^{'}(x)$ and Class 2: $D^{\alpha}f(x) = \cos\left(\frac{\alpha\pi}{2}\right)x^{\alpha}f(x) + \sin\left(\frac{\alpha\pi}{2}\right)x^{1-\alpha}f^{'}(x)$) of the modified conformable differential operator yield identical results for the eigenvalues and eigenfunctions as the standard derivative when $\alpha = 1$, as shown in Tables I and II and Figs. 14–16 and Figs. 17–19.
- For fractional values of α, varying the classes of the given operator produces different results for λ_n and $u_n(x)$, which can be observed in Tables I and II and Figs. 14–22.

TABLE II. EIGENVALUES AND CORRESPONDING EIGENFUNCTIONS FOR VARYING α AND n

α	n	$RA(\lambda_n)$	$NV(\lambda_n)$	$u_n(x)$
1	1	6.47	6.467401100	$e^{2x}\sin\left(\frac{\pi x}{2}\right)$
1	50	$\frac{1117223}{181}$	6172.502751	$e^{2x}\sin(25\pi x)$
1	100	$\frac{2221021}{90}$	24678.01100	$e^{2x}\sin(50\pi x)$
$\frac{1}{2}$	1	4.62	4.616850275	$e^{(4\sqrt{2}\sqrt{x}-x)}\sin\left(\frac{\pi\sqrt{x}}{\sqrt{2}}\right)$
$\frac{1}{2}$	50	$\frac{1}{22}\left(16995+\sqrt{289672405}\right)$	1546.125688	$e^{(4\sqrt{2}\sqrt{x}-x)}\sin\left(25\sqrt{2}\pi\sqrt{x}\right)$
$\frac{1}{2}$	100	$\frac{1117223}{181}$	6172.502751	$e^{(4\sqrt{2}\sqrt{x}-x)}\sin\left(50\sqrt{2}\pi\sqrt{x}\right)$
$\frac{1}{20}$	1	$\frac{28225}{7056}$	4.000141717	$e^{\left(-10x^{\frac{1}{20}}\left(-4+x^{\frac{1}{20}}\cos\left(\frac{\pi}{40}\right)\right)\csc\left(\frac{\pi}{40}\right)\right)}\sin\left(\frac{\pi x^{\frac{1}{20}}}{2^{\frac{1}{20}}}\right)$
$\frac{1}{20}$	50	4.35	4.354293643	$e^{\left(-10x^{\frac{1}{20}}\left(-4+x^{\frac{1}{20}}\cos\left(\frac{\pi}{40}\right)\right)\csc\left(\frac{\pi}{40}\right)\right)}\sin\left(25\times 2^{\frac{19}{20}}\pi x^{\frac{1}{20}}\right)$
$\frac{1}{20}$	100	$\frac{1}{16}\left(1141+\sqrt{1417241}\right)$	145.7174572	$e^{\left(-10x^{\frac{1}{20}}\left(-4+x^{\frac{1}{20}}\cos\left(\frac{\pi}{40}\right)\right)\csc\left(\frac{\pi}{40}\right)\right)}\sin\left(50\times 2^{\frac{19}{20}}\pi x^{\frac{1}{20}}\right)$

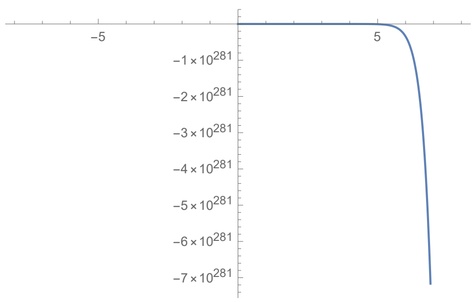

Fig. 11. $u_n(x)$ with $\alpha = \frac{1}{20}$ and $n = 1$

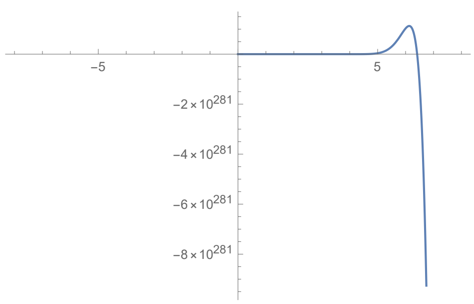

Fig. 12. $u_n(x)$ with $\alpha = \frac{1}{20}$ and $n = 50$

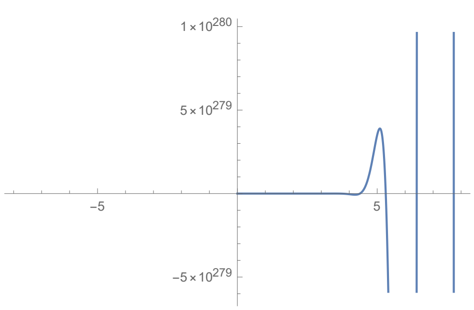

Fig. 13. $u_n(x)$ with $\alpha = \frac{1}{20}$ and $n = 100$

- For the classical case $\alpha = 1$ and the fractional case, the coefficient n affects the values of the eigenvalues λ_n, as shown in the tabulated values in Tables I and II. Larger values of n yield larger eigenvalues λ_n, as seen in Fig. 15, Fig. 16, Fig. 18, and Fig. 19.
- The fractional order α affects the values of the eigenvalues λ_n, as highlighted by comparing the numerical values in Tables I and II, and the behavior of the eigenfunctions $u_n(x)$ illustrated in Figs. 14–22.
- For the classical case $\alpha = 1$, as n increases, the eigenfunctions $u_n(x)$ become more oscillatory, as evident from Fig. 15 and Fig. 16.
- For the fractional case, as α decreases, the eigenfunctions $u_n(x)$ exhibit slower oscillations and become more stable, as demonstrated in Figs. 20–22.

Problem 2. Let $\kappa_0(\alpha, x), \kappa_1(\alpha, x) : [0,1] \times \mathbb{R} \to [0, \infty)$ be continuous and satisfy (2) such that $\kappa_1(\alpha, x)$ is a real constant. Our objective is to determine the eigenvalues and the corresponding eigenfunctions for the following modified conformable Sturm-Liouville problem:

$$D^\alpha D^\alpha u(x) - \lambda u(x) = 0, \quad 0 < x < L, \quad u(0) = u(L) = 0. \tag{43}$$

First, we will write the characteristic equation for the above problem:

$$r^2 - \lambda = 0,$$

which has the roots:

$$r = \pm\sqrt{\lambda}.$$

Case 1: If $\lambda > 0$, then we have real distinct roots given by $r = \pm\sqrt{\lambda}$, and by the Modified Conformable Constant Coefficients Theorem, the solution is given by:

$$u(x) = c_1 e_{+\sqrt{\lambda}}(x, 0) + c_2 e_{-\sqrt{\lambda}}(x, 0),$$

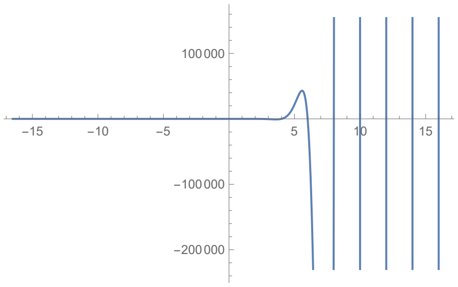

Fig. 14. $u_n(x)$ with $\alpha = 1$ and $n = 1$

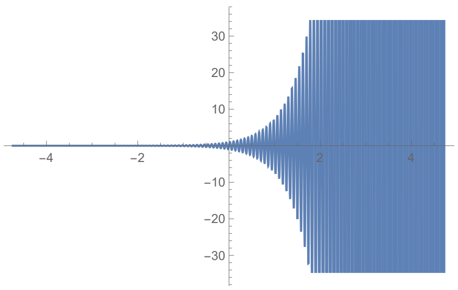

Fig. 15. $u_n(x)$ with $\alpha = 1$ and $n = 50$

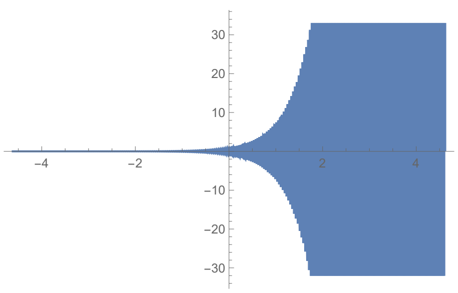

Fig. 16. $u_n(x)$ with $\alpha = 1$ and $n = 100$.

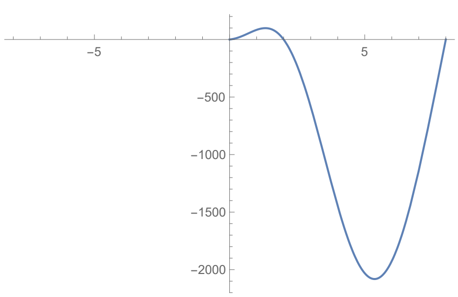

Fig. 17. $u_n(x)$ with $\alpha = \frac{1}{2}$ and $n = 1$.

where c_1 and c_2 are constants. Using the boundary conditions, we obtain:

$$\begin{aligned} u(0) &= \left(c_1 e_{+\sqrt{\lambda}}(x,0) + c_2 e_{-\sqrt{\lambda}}(x,0)\right)\Big|_{x=0} \\ &= c_1 e_{+\sqrt{\lambda}}(0,0) + c_2 e_{-\sqrt{\lambda}}(0,0) \\ &= c_1 + c_2 = 0. \end{aligned}$$

Thus, we have $c_2 = -c_1$. Now, applying the second boundary condition:

$$\begin{aligned} u(L) &= \left(c_1 e_{+\sqrt{\lambda}}(x,0) + c_2 e_{-\sqrt{\lambda}}(x,0)\right)\Big|_{x=L} \\ &= \left(c_1 e_{+\sqrt{\lambda}}(L,0) - c_1 e_{-\sqrt{\lambda}}(L,0)\right) \\ &= c_1 \left(e_{+\sqrt{\lambda}}(L,0) - e_{-\sqrt{\lambda}}(L,0)\right) \\ &= 0. \end{aligned}$$

Since it is evident that

$$e_{+\sqrt{\lambda}}(L,0) - e_{-\sqrt{\lambda}}(L,0) \neq 0,$$

we conclude that $c_1 = c_2 = 0$. Therefore, λ in this case is not an eigenvalue of the problem.

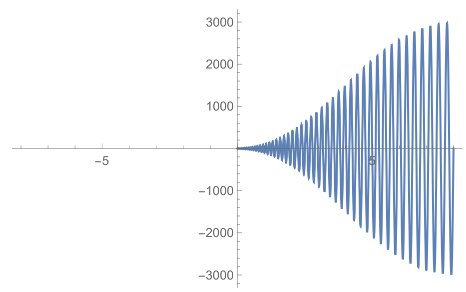

Fig. 18. $u_n(x)$ with $\alpha = \frac{1}{2}$ and $n = 50$.

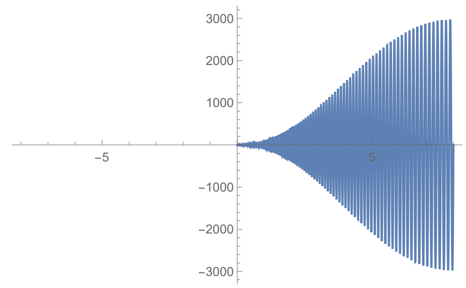

Fig. 19. $u_n(x)$ with $\alpha = \frac{1}{2}$ and $n = 100$.

Case 2: If $\lambda = 0$, then we have real repeated roots $r = 0$. By the Modified Conformable Constant Coefficients Theorem, the solution is given by:

$$u(x) = c_1 e_0(x,0) + c_2 e_0(x,0) \int_0^x 1 d_\alpha s,$$

where c_1 and c_2 are constants. Applying the boundary conditions:

$$\begin{aligned} u(0) &= \left(c_1 e_0(x,0) + c_2 e_0(x,0) \int_0^x 1 d_\alpha s\right)\Big|_{x=0} \\ &= c_1 e_0(0,0) + c_2 e_0(0,0) \int_0^0 1 d_\alpha s \\ &= c_1 = 0. \end{aligned}$$

For the second boundary condition:

$$\begin{aligned} u(L) &= \left(c_1 e_0(x,0) + c_2 e_0(x,0) \int_0^x 1 d_\alpha s \right) \Big|_{x=L} \\ &= c_2 e_0(L,0) \int_0^L 1 d_\alpha s = 0. \end{aligned}$$

It is evident that

$$e_0(L,0) \int_0^L 1 d_\alpha s \neq 0.$$

Therefore, we conclude that $c_1 = c_2 = 0$, and hence, λ in this case is not an eigenvalue of the problem.

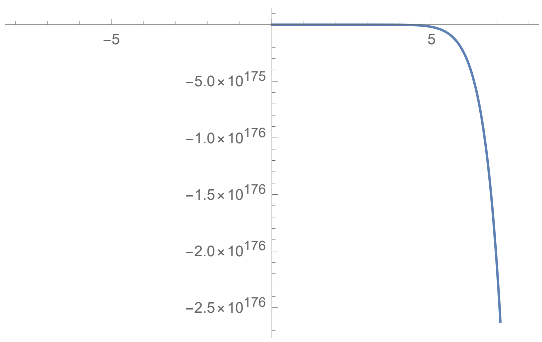

Fig. 20. $u_n(x)$ with $\alpha = \frac{1}{20}$ and $n = 1$.

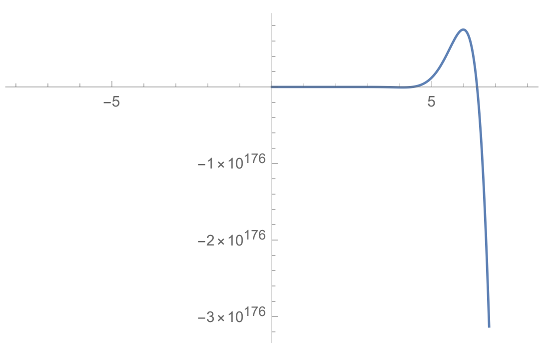

Fig. 21. $u_n(x)$ with $\alpha = \frac{1}{20}$ and $n = 50$.

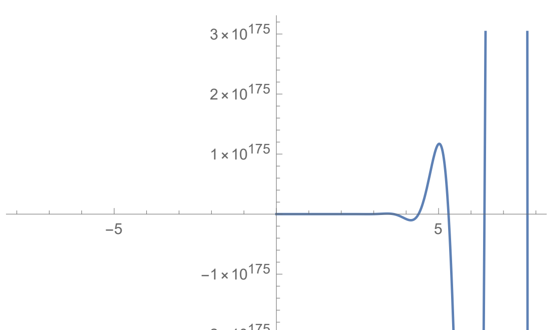

Fig. 22. $u_n(x)$ with $\alpha = \frac{1}{20}$ and $n = 100$.

Case 3: If $\lambda < 0$, then we have complex roots $r = \pm i\sqrt{|\lambda|}$. According to the Modified Conformable Constant Coefficients Theorem, the solution is given by:

$$\begin{aligned} u(x) = c_1 e_0(x,0) \cos\left(\int_0^x \sqrt{|\lambda|} d_\alpha s \right) \\ + c_2 e_0(x,0) \sin\left(\int_0^x \sqrt{|\lambda|} d_\alpha s \right), \end{aligned}$$

where c_1 and c_2 are constants. Applying the boundary conditions:

$$\begin{aligned} u(0) = &\left(c_1 e_0(x,0) \cos\left(\int_0^x \sqrt{|\lambda|} d_\alpha s \right) \right. \\ &\left. + c_2 e_0(x,0) \sin\left(\int_0^x \sqrt{|\lambda|} d_\alpha s \right) \right) \Big|_{x=0} \\ = &\left(c_1 e_0(0,0) \cos\left(\int_0^0 \sqrt{|\lambda|} d_\alpha s \right) \right. \\ &\left. + c_2 e_0(0,0) \sin\left(\int_0^0 \sqrt{|\lambda|} d_\alpha s \right) \right) \\ = &\; c_1 = 0. \end{aligned}$$

For the second boundary condition:

$$\begin{aligned} u(L) = &\left(c_1 e_0(x,0) \cos\left(\int_0^x \sqrt{|\lambda|} d_\alpha s \right) \right. \\ &\left. + c_2 e_0(x,0) \sin\left(\int_0^x \sqrt{|\lambda|} d_\alpha s \right) \right) \Big|_{x=L} \\ = &\; c_2 e_0(L,0) \sin\left(\int_0^L \sqrt{|\lambda|} d_\alpha s \right) \\ = &\; 0. \end{aligned}$$

From this, we get:

$$c_2 \sin\left(\int_0^L \sqrt{|\lambda|} d_\alpha s \right) = 0.$$

Assuming that $c_2 \neq 0$, we obtain:

$$\sin\left(\int_0^L \sqrt{|\lambda|} d_\alpha s \right) = 0.$$

This implies:

$$\int_0^L \sqrt{|\lambda|} d_\alpha s = n\pi, \quad n \in \mathbb{N}.$$

Therefore, we conclude that the eigenvalues are:

$$\lambda_n = \left(\frac{n\pi}{\int_0^L 1 d_\alpha s} \right)^2, \quad n \in \mathbb{N}.$$

The corresponding eigenfunctions associated with the eigenvalues λ_n are given by:

$$u_n(x) = e_0(x,0) \sin\left(\frac{n\pi \int_0^x 1 d_\alpha s}{\int_0^L 1 d_\alpha s} \right), \quad n \in \mathbb{N}.$$

The analytic solution of Problem 2 shows that the eigenfunctions and eigenvalues can assume different values depending on the fractional order α.

IV. Conclusion and Future Work

In this study, we investigated the modified conformable self-adjoint equation and fractional Sturm-Liouville problems. The key conclusions are summarized as follows:

- The modified conformable Sturm-Liouville equation is a second-order linear homogeneous differential equation of the form:

$$D^{\alpha}\left[p(x)\left(D^{\alpha}u(x)-\kappa_1(\alpha,x)u(x)\right)\right]+(q(x)+\lambda r(x))u(x)=0.$$

- The corresponding modified conformable Sturm-Liouville operator is defined as:

$$L=D^{\alpha}\left[p(x)\left(D^{\alpha}-\kappa_1(\alpha,x)\right)\right]+q(x).$$

- The operator L is self-adjoint if:

$$\int_a^b f(Lg)d_{\alpha}x=\int_a^b g(Lf)d_{\alpha}x.$$

- The Sturm-Liouville equation is considered regular if:

$$p(x)>0 \text{ and } q(x)>0 \quad \forall x\in[a,b].$$

- A periodic modified conformable Sturm-Liouville system satisfies the boundary conditions:

$$u(a)=u(b) \text{ and } D^{\alpha}u(a)=D^{\alpha}u(b).$$

- The system is singular if:

$$p(x)>0 \text{ on } (a,b),\ r(x)\geq 0 \text{ on } [a,b], \text{ and } p(a)=p(b)=0.$$

Future research will focus on a detailed investigation of the classifications mentioned above, along with the following directions:

- Exploring the applicability of the Rayleigh quotient for estimating eigenvalues.
- Analyzing eigenfunction expansions and the Fredholm alternative theorem using the modified conformable operator.
- Extending the developed framework to higher-order fractional differential equations.

The modified conformable operator provides a robust framework for fractional calculus, offering improved computational efficiency and mathematical consistency. Potential applications include:

- **Physics:** Modeling anomalous diffusion and viscoelastic materials.
- **Engineering:** Applications in signal processing, control systems, and materials science.
- **Mathematical Biology:** Describing memory-dependent biological processes.
- **Spectral Theory:** Analyzing eigenvalue problems in quantum mechanics and wave propagation.

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